WED 15 JAN 2025

As usual we work on the time interval $[0,T]$ for fixed $T>0$.

Given a GERM $A=(A_{st})_{0\le s\le t\le T}$, we look for a corresponding INTEGRAL $I=(I_t)_{0\le t\le T}$ which satisfies $I_0=0$ (say) and

$$\tilde{\star} \qquad \delta I_{st} = A_{st} + \underbrace{o(t-s)}_{R_{st} := \delta I_{st} - A_{st}}$$

REMINDER

If I exists, then it is UNIQUE (for a fixed germ A)

For existence of I, it is necessary that $\delta A_{sut} = o(t-s)$.

Remarkably, it is sufficient that $\delta A_{sut} = O((t-s)^\eta)$ for some $\eta>1$

$\iff \|\delta A\|_\eta < \infty$ (η-COHERENCE)

SEWING LEMMA: $\exists\, \eta>1: \|\delta A\|_\eta<\infty \Rightarrow \exists\, I=(I_t)$ s.t. $\tilde{\star}$ holds.

$\Downarrow$

SEWING BOUND: $R_{st} = O((t-s)^\eta)$: $\|R\|_\eta \le K_\eta \|\delta A\|_\eta$

$\delta A_{sut} = -\delta Y_{su}\,\delta X_{ut}$

YOUNG INTEGRAL: $A_{st} = Y_s\,\delta X_{st}$ $\qquad X\in\mathcal{C}^\alpha$, $Y\in\mathcal{C}^\beta$

The germ A is $\eta=(\alpha+\beta)$-COHERENT: if $\alpha+\beta>1 \Rightarrow \exists\, I$:

$$\star' \qquad \delta I_{st} = Y_s\,\delta X_{st} + \underbrace{o(t-s)}_{R_{st} = O((t-s)^{\alpha+\beta})} \qquad \text{“} I_t = \int_0^t Y_u\, dX_u \text{”}$$

Henceforth we focus on the regime $\alpha+\beta \leq 1$ (ROUGH CASE)

In this regime $(\star')$ has, in general, no solution!

Indeed, a necessary condition for the existence of a solution is $\delta A_{sut} = o(t-s)$. But for $A_{st} = Y_s \, \delta X_{st}$

$$\delta A_{sut} = -\delta Y_{su} \, \delta X_{ut}$$

For $X \in \mathcal{C}^\alpha$, $Y \in \mathcal{C}^\beta$ we have $|\delta A_{sut}| = O((t-s)^{\alpha+\beta})$ but, in general, not better! Indeed

$$X_t = t^\alpha, \quad Y_t = t^\beta \quad \Rightarrow \delta A_{0,t,2t} = -t^\beta \cdot t^\alpha = -t^{\alpha+\beta} \neq o(2t-0) \; !$$

The idea is to relax $(\star')$ replacing $o(t-s)$ by $O((t-s)^\gamma)$ for some $\gamma \leq 1$. By the previous example, a natural choice is $\gamma = \alpha+\beta$.

Def. Given $\alpha, \beta \in (0,1]$ and $X \in \mathcal{C}^\alpha$, $Y \in \mathcal{C}^\beta$, we call a generalized $(\alpha+\beta)$-integral of Y w.r.t. X any function $I = (I_t)_{t \in [0,T]}$ s.t. $I_0 = 0$ (say) and

$(\star'')$ $$\delta I_{st} = Y_s \, \delta X_{st} + O((t-s)^{\alpha+\beta}).$$

Remark. If $\alpha+\beta\le 1$, we never have uniqueness: indeed, if I solves ($\star''$), also $\tilde{I} := I + f$ with $f \in \mathcal{C}^{\alpha+\beta}$ solves ($\star''$), just because

$$\delta\tilde{I}_{st} = \delta I_{st} + \underbrace{\delta f_{st}}_{f_t - f_s = O((t-s)^{\alpha+\beta})}$$

In fact, this describes ALL SOLUTIONS of ($\star''$): given any two solutions $I, \tilde{I}$ of ($\star''$), their difference $f := \tilde{I} - I$ satisfies

$$\delta f_{st} = \underbrace{\delta\tilde{I}_{st} - \delta I_{st}}_{Y_s \delta X_{st} + O((t-s)^{\alpha+\beta})} = O((t-s)^{\alpha+\beta}) \quad \text{i.e. } f \in \mathcal{C}^{\alpha+\beta}.$$

Theorem (LYONS - VICTOIR). For any $\alpha, \beta \in (0,1]$ and any $X \in \mathcal{C}^\alpha$, $Y \in \mathcal{C}^\beta$, there is a solution I of ($\star''$), i.e. I is an $(\alpha+\beta)$-integral of Y w.r.t. X.

(We are assuming that X, Y are real valued!)

Remark. In the special case $\beta = \alpha$ and $X = Y$, we have a natural, explicit choice of (2α)-integral I of X w.r.t. X, namely $I_t := \frac{X_t^2}{2}$.

$b^2 = (a + (b-a))^2$
$\overset{!}{=} a^2 + 2a(b-a) + (b-a)^2$

$a = X_s$
$b = X_t$

Indeed $\quad \delta I_{st} = I_t - I_s = \dfrac{X_t^2 - X_s^2}{2}$

$$\overset{!}{=} \frac{2X_s(X_t - X_s) + (X_t - X_s)^2}{2}$$

$$\overset{!}{=} X_s \cdot (X_t - X_s) + \underbrace{\tfrac{1}{2}(X_t - X_s)^2}_{O((t-s)^{2\alpha})}$$

Therefore ANY (2α)-integral of X w.r.t. X is given by

$$I_t = \frac{X_t^2}{2} + f_t \qquad \text{for some } f \in \mathcal{C}^{2\alpha}.$$

Let us now revisit the concept of <u>rough paths</u>.

Fix a path $X : [0,T] \to \mathbb{R}^d$ of class $\mathcal{C}^\alpha$ with $\frac{1}{3} < \alpha \le \frac{1}{2}$.
$X = (X^1, \dots, X^d)$.

In order to define a notion of integral of the path w.r.t. itself, we apply the definition above:
$\forall\, i,j \in \{1, \dots, d\}$ we fix an 2α-integral I^{ij} of X^i w.r.t. X^j, that is $I^{ij}_0 = 0$ (say) and

$$(\star) \qquad \delta I^{ij}_{st} = X^i_s\, \delta X^j_{st} + O((t-s)^{2\alpha})$$

Let us call the remainder

$$\boxtimes \qquad (\mathbb{X}^2)^{ij}_{st} := \delta I^{ij}_{st} - X^i_s \, \delta X^j_{st}.$$

Lemma. If I^{ij} is a 2α-integral of X^i w.r.t. X^j, then $(\mathbb{X}^2)^{ij}$ satisfies

$$(𝄞) \qquad (\mathbb{X}^2)^{ij}_{st} = O\big((t-s)^{2\alpha}\big) \qquad \overbrace{(\delta \mathbb{X}^2)^{ij}_{sut} = \delta X^i_{su} \, \delta X^j_{ut}}^{\text{CHEN RELATION}}$$

Vice versa, given any function $(\mathbb{X}^2)^{ij}_{st}$ which satisfies (𝄞), there exists a unique I^{ij} which is a (2α)-integral of X^i w.r.t. X^j s.t. $\boxtimes$ holds.

Proof. Exercise!

In view of the previous result, in order to define an integral of X^i w.r.t. X^j, it is equivalent to assign a "reminder" $\mathbb{X}^2$ which satisfies (𝄞). This leads to the notion of ROUGH PATH that we already gave and that we recall.

Def. Given $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right]$ and a path $X \in \mathcal{C}^\alpha(\mathbb{R}^d)$, we call α-ROUGH PATH over X any $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$ with $\mathbb{X}^1 = \left((\mathbb{X}^1)^i_{st}\right)_{\substack{i=1,\dots,d \\ 0 \le s \le t \le T}}$ and $\mathbb{X}^2 = \left((\mathbb{X}^2)^{ij}_{st}\right)_{\substack{i,j=1,\dots,d \\ 0 \le s \le t \le T}}$

(i.e. $\mathbb{X}^1 : [0,T]^2_{\leq} \to \mathbb{R}^d$, $\mathbb{X}^2 : [0,T]^2_{\leq} \to \mathbb{R}^d \otimes \mathbb{R}^d$)

s.t. the following holds:

(i) $\mathbb{X}^1_{st} = \delta X_{st}$ $\qquad$ $\delta \mathbb{X}^2_{sut} = \mathbb{X}^1_{su} \otimes \mathbb{X}^1_{ut}$

$$(\delta \mathbb{X}^2)^{ij}_{sut} = (\mathbb{X}^1)^i_{su} \, (\mathbb{X}^1)^j_{ut}$$

(ii) $|\mathbb{X}^1_{st}| = O((t-s)^\alpha)$ $\qquad$ $|\mathbb{X}^2|_{st} = O((t-s)^{2\alpha})$

i.e. $\|\mathbb{X}^1\|_\alpha < \infty$ $\qquad$ $\|\mathbb{X}^2\|_{2\alpha} < \infty$

Keep in mind: $(\mathbb{X}^2)^{ij}_{st} = I^{ij}_{st} - X^i_s \, \delta X^j_{st}$

$$I^{ij}_{st} = \text{"} \int_s^t (X^i_u - X^i_s) \, dX_u \text{"}$$

Henceforth we fix a path $X \in \mathcal{C}^\alpha$, with $\frac{1}{3} < \alpha \leq \frac{1}{2}$,
and a α-ROUGH PATH $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$ over X.

How to define a notion of integral

$$\text{"} \int_0^t Y_u \, dX_u \text{"}$$

for a reasonably large class of paths $Y = (Y_u)$?

We assume that $Y : [0,T] \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^k)$.

The idea is that, since we already know how to integrate X w.r.t. X (i.e. X^i w.r.t. X^j $\forall i,j$), thanks to the rough path $\mathbb{X}$ that we have fixed, we may hope to be able to give a CANONICAL notion of integral for all paths Y which locally "look like X". Let us make the latter notion precise.

Def. (CONTROLLED PATHS). Fix $\ell \in \mathbb{N}$. Fix $\alpha \in (\frac{1}{3}, \frac{1}{2}]$ and a path $X \in \mathcal{C}^\alpha(\mathbb{R}^d)$ (actually its increments $\mathbb{X}^1_{st} = \delta X_{st}$). Also fix $\eta \in (0,1]$. We call $(\alpha+\eta)$-PATH CONTROLLED BY X any pair (Y, Y') where $Y : [0,T] \to \mathbb{R}^\ell$ and $Y' : [0,T] \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^\ell)$ such that $Y \in \mathcal{C}^\alpha$, $Y^1 \in \mathcal{C}^\eta$ and

$$\delta Y_{st} = Y^1_s \, \delta X_{st} + O((t-s)^{\alpha+\eta})$$

We call Y^1 a derivative of Y w.r.t. X.
It is also useful to introduce the "remainder"

$$Y^{[2]}_{st} := \delta Y_{st} - Y^1_s \, \delta X_{st} = O((t-s)^{\alpha+\eta})$$

We can finally state our main result for today.

<u>Theorem</u>. Fix $d, k \in \mathbb{N}$, Fix $\alpha \in (\frac{1}{3}, \frac{1}{2}]$ and a α-rough path $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$. Fix $\eta \in (0,1]$ and a $(\alpha+\eta)$-controlled path $\mathbf{Y} = (Y, Y')$ where
$Y : [0,T] \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^k) \simeq \mathbb{R}^{d \times k}$ and
$Y' : [0,T] \to \mathcal{L}(\mathbb{R}^d \otimes \mathbb{R}^d, \mathbb{R}^k) \simeq \underbrace{\mathcal{L}(\mathbb{R}^d, \mathcal{L}(\mathbb{R}^d, \mathbb{R}^k))}_{\simeq \mathbb{R}^{d \times d \times k}}$.

Then the following germ

$$A_{st} = Y_s \mathbb{X}^1_{st} + Y'_s \mathbb{X}^2_{st}$$

is $(2\alpha+\eta)$-COHERENT. Therefore, if $2\alpha+\eta > 1$, the exists a unique "integral" $I = (I_t)_{t \in [0,T]}$ of the germ A, that is I satisfies $I_0 = 0$ and

$$\delta I_{st} = \underbrace{Y_s \mathbb{X}^1_{st} + Y'_s \mathbb{X}^2_{st}}_{A_{st}} + O\left((t-s)^{2\alpha+\eta}\right)$$

We call I the ROUGH INTEGRAL of the controlled path $\mathbf{Y}$ w.r.t. the rough path $\mathbb{X}$.

We may write $I_t = \int_0^t \mathbf{Y} \, d\mathbb{X}$ or $I_t = \int_0^t Y_u \, dX_u$.

Finally, for any $t \in [0,T]$ we have

$$I_t = \lim_{|\mathcal{P}| \to 0} \sum_{i=0}^{\#\mathcal{P}-1} \left(Y_{t_i} \mathbb{X}^1_{t_i t_{i+1}} + Y^1_{t_i} \mathbb{X}^2_{t_i t_{i+1}} \right)$$

<u>Proof</u> We only need to check the coherence of A:

$$A_{st} = Y_s \underbrace{\mathbb{X}^1_{st}}_{\delta X_{st}} + Y^1_s \mathbb{X}^2_{st}$$

$$\delta A_{sut} = -\delta Y_{su} \mathbb{X}^1_{ut} - \delta Y^1_{su} \mathbb{X}^2_{ut} + Y^1_s \underbrace{\delta \mathbb{X}^2_{sut}}_{\mathbb{X}^1_{su} \otimes \mathbb{X}^1_{ut}}$$

$$= -\Big(\underbrace{\delta Y_{su} - Y^1_s \mathbb{X}^1_{su}}_{Y^{[2]}_{su} = O((u-s)^{\alpha+\eta})} \Big) \underbrace{\mathbb{X}^1_{ut}}_{O((t-u)^{\alpha})} - \underbrace{\delta Y^1_{su}}_{O((u-s)^{\eta})} \underbrace{\mathbb{X}^2_{ut}}_{O((t-u)^{2\alpha})}$$

$$= O\left((t-s)^{2\alpha+\eta} \right)$$ i.e. A is $(2\alpha+\eta)$-coh.

Motivation: how to define $I_t = \int_0^t Y_u \, dX_u$?

$$\delta I_{st} = I_t - I_s = \int_s^t Y_u \, dX_u = Y_s \cdot \delta X_{st} + \int_s^t (Y_u - Y_s) \, dX_u$$

$$= Y_s \cdot \delta X_{st} + \int_s^t Y^1_s \, \delta X_{su} \, dX_u + \int_s^t Y^{[2]}_{su} \, dX_u$$

Thus $\delta I_{st} = \overbrace{Y_s \, \delta X_{st} + Y'_s \, \mathbb{X}^2_{st}}^{A_{s,t}} + R_{st}$

with $R_{st} = " \int_s^t \underbrace{Y^{[2]}_{su}}_{\alpha+\eta} \, \underbrace{dX_u}_{\alpha} "$

Hope: $R_{st} = O((t-s)^{2\alpha+\eta})$? ..

Let us conclude. Given an α-ROUGH PATH $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$ and a $(\alpha+\eta)$-CONTROLLED PATH $\mathbf{Y} = (Y, Y')$, we defined the ROUGH INTEGRAL $I = (I_t)_{t \in [0,t]}$, assuming $2\alpha + \eta > 1$.

By construction $\delta I_{st} = Y_s \, \delta X_{st} + Y^1_s \, \underbrace{\mathbb{X}^2_{st}}_{O((t-s)^{2\alpha})} + O((t-s)^{2\alpha+\eta})$

$$= Y_s \, \delta X_{st} + O((t-s)^{2\alpha})$$

which means that the pair $\mathbf{I} = (I, I^1 = Y)$ is a

(2α)-CONTROLLED PATH. Let us set

$$\mathbf{I} = \int \mathbf{Y} \, d\mathbb{X}$$

If we fix from the beginning $\eta = \alpha$, i.e. we fix a 2α-controlled path $\mathbf{Y} = (Y, Y')$, then assuming $2\alpha + \eta = 3\alpha > 1$, i.e. $\alpha > \frac{1}{3}$, we have that also the (enriched) rough integral $\mathbb{I} = (I, I^1 = Y)$ is a (2α)-controlled path. Denoting by

$$\mathcal{D}_X^{2\alpha} := \{ 2\alpha\text{-CONTROLLED PATHS BY } X \}$$

The rough integral defines a map from $\mathcal{D}_X^{2\alpha}$ to itself.

We note that $\mathcal{D}_X^{2\alpha}$ is a LINEAR SPACE which becomes a BANACH SPACE equipped with the norm

for $\mathbb{Y} = (Y, Y')$:

$$\| \mathbb{Y} \|_{\mathcal{D}_X^{2\alpha}} := |Y_0| + |Y'_0| + [\mathbb{Y}]_{\mathcal{D}_X^{2\alpha}}$$

$$[\mathbb{Y}]_{\mathcal{D}_X^{2\alpha}} := \| \delta Y' \|_\alpha + \| Y^{[2]} \|_{2\alpha}$$

We can show that the rough integral $\mathbb{Y} \mapsto \mathbb{I}$ is a CONTINUOUS MAP ON $\mathcal{D}_X^{2\alpha}$.
Even more, the joint map $(\mathbb{X}, \mathbb{Y}) \mapsto \mathbb{I}$ is continuous, in fact locally Lipschitz. This lets one solve integral equations driven by rough paths by usual thechinques, i.e. fixed point theorems.

Recall the starting differential equation:

$$\dot{Z}_t = \sigma(Z_t) \dot{X}_t$$

that we interpret as an integral equation

$$\mathbf{Z}_t - \mathbf{Z}_0 = \mathbf{I}_t(\sigma(\mathbf{Z}), \mathbb{X}) \tag{✿}$$

where $\mathbf{Z}_t = (Z_t, Z_t^1) \in \mathcal{D}_X^{2\alpha}$

and $\sigma(\mathbf{Z}_t) := (\sigma(Z_t), \nabla\sigma(Z_t)\cdot Z_t^1)$

Note that if $\mathbf{Z}$ is a solution of ✿, then

$$Z_t^1 = \sigma(Z_t)$$

hence $\sigma(\mathbf{Z}_t) = (\sigma(Z_t), \underbrace{\nabla\sigma(Z_t)\cdot\sigma(Z_t)}_{\sigma_2(Z_t)})$.