Invariance Principles for lifted random processes

0) **Motivation**

Consider $(X_n)_{n \in \mathbb{N}_0}$ RW on $\mathbb{R}^d$

$(\xi_k = X_k - X_{k-1})_{k \in \mathbb{N}}$ i.i.d s.t. $\forall i, j \in \{1, \dots, d\}$:

$$\mu^i = \mathbb{E}[\xi_1^i] = 0 \, , \; \Sigma^{ij} := \mathbb{E}[\xi_1^i \xi_1^j] \in \mathbb{R}$$

Theorem (Donsker I.P.)

Fix $T > 0$. consider two rescaling of X:

(i) $X_t^{(n)} := \frac{1}{\sqrt{n}} X_{\lfloor nt \rfloor}$

(ii) $\overline{X}_t^{(n)} := \frac{1-w}{\sqrt{n}} X_k + \frac{w}{\sqrt{n}} X_{k+1}$; $k := \lfloor nt \rfloor$, $w := nt - \lfloor nt \rfloor$

Notation: $(Z_k)_{k\in\mathbb{N}}$

$\leadsto$ $Z_\cdot^{(n)}, \bar{Z}_\cdot^{(n)} : [0,T] \to \mathbb{R}^d$ defined as $X^{(n)}, \bar{X}^{(n)}$.

Then ① $(\bar{X}_t^{(n)})_{[0,T]} \xrightarrow{\text{law}} (\beta_t)_{[0,T]}$

in $C([0,T], \mathbb{R}^d)$, where

w/ uniform topology

$\beta_t \equiv \Sigma^{1/2} B_t$

$\Sigma = \Sigma^{1/2}(\Sigma^{1/2})^T$

$\mathbb{E}[\beta_t^i \beta_t^j] = t\Sigma^{ij}$

② $(X_t^{(n)})_{[0,T]} \xrightarrow{\text{law}} (\beta_t)_{[0,T]}$

in $D([0,T], \mathbb{R}^d)$

w/ Skorohod topology

Theorem (Wong-Zakai 1965)

Under the same conditions as in class of 18/12/2024

Look at

(1) $$\bar{Y}_t^{(n)} = \bar{Y}_0^{(n)} + \int_0^t \sigma(\bar{Y}_s^{(n)})\, d\bar{X}_s^{(n)}$$

Riemann-Stieltjes

(2) $$Y_t^{(n)} = Y_0^{(n)} + \int_0^t \sigma(Y_s^{(n)})\, dX_s^{(n)}.$$

left point sum

Then $\bar{Y}^{(n)} \xrightarrow{\text{law}} \bar{Y}$ in C,

$Y^{(n)} \xrightarrow{\text{law}} Y$ in D, where $\tilde{Y}$ solves (Stratonovich SDE)

Y solves (Itô SDE)

Remember

If $\mathbb{X}$ and $\tilde{\mathbb{X}}$ are two α-r.p. over X for $\alpha \in (\frac{1}{3}, \frac{1}{2}]$, then

$\tilde{\mathbb{X}}^{(2)}_{s,t} = \mathbb{X}^{(2)}_{s,t} + \delta f_{s,t}$ where $f \in C^{\alpha}$.

General Brownian Rough Path

For $\Sigma, \Gamma \in \mathbb{R}^{d\times d}$, Σ symmetric.

$\mathbb{B}$ is (Σ, Γ) B.r.p if

$$\mathbb{B}^{(1)}_{0,t} = \Sigma^{1/2} B_{0,t} =: \beta_t \text{ and}$$

$$\mathbb{B}^{(2)}_{s,t} = \int_s^t \beta_{s,u} \otimes d\beta_u + (t-s)\Gamma$$

Itô.

<u>Notation:</u> $X_{s,t} := \delta X_{s,t} = X_t - X_s$

<u>Examples</u>

Itô B.r.p. is $(I, 0)$-B.r.p

Stratonovich B.r.p. is $(I, \frac{1}{2} I)$-B.r.p.

<u>Theorem</u> (Kelly-Melbourne '16, Kelly '16)

$\sigma: \mathbb{R}^k \to \mathbb{R}^{k \times d}$

$X^{(n)}: [0,T] \longrightarrow \mathbb{R}^d$

Assume that σ, $\bar{X}^{(n)}$ are "nice",

If $\mathbb{X}^{(n)} \xrightarrow{\text{law}} \mathbb{B}$ in C^α $\alpha \in (\frac{1}{3}, \frac{1}{2}]$

defined as above "stratonovich" iterated integral of $X^{(n)}$

$\parallel$

$(I, \frac{1}{2}I + \Gamma)$ B.r.p,

then $Y^{(n)} \xrightarrow{\text{law}} Y$ in C^α, where

$Y^{(n)}$ sol. to Stratonovich SDE

driven by $X^{(n)}$. Moreover

Y solves

$$Y_t = Y_0 + \int_0^t \sigma(Y_s) \circ dB_s + \int_0^t K(\sigma,\Gamma)(Y_s)\, ds,$$

where $K(\sigma,\Gamma) = D\sigma(y)\,\sigma(y)\,\Gamma$

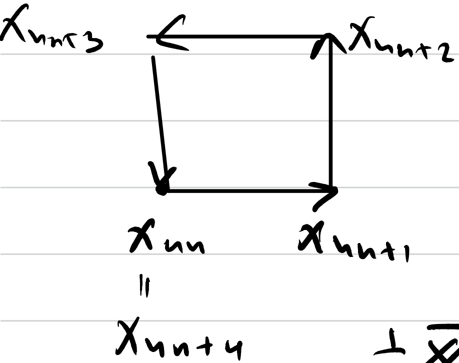
<u>Goal</u> Study how Γ appears in natural models

<u>Example 1</u>

Let (X_n) be a process on $\mathbb{Z}^2$ defined by $X_0 = (0,0)$, $X_1 = (1,0)$

$X_2 = (1,1)$, $X_3 = (0,1)$

$$X_{4n+j} = X_j \quad \forall n \in \mathbb{N},\ j \in \{0,1,2,3\}$$



$\bar{X}^{(n)}$, $\bar{\mathbb{X}}^{(n)}$ as before

$$\|\bar{X}^{(n)}_{\cdot}\|_\infty \le \frac{1}{\sqrt{n}} \longrightarrow 0$$

$$\frac{1}{n}\bar{\mathbb{X}}_{0,4n} = \frac{c}{n}\begin{pmatrix} 0 & n \\ -n & 0 \end{pmatrix} = c\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\left(\bar{X}^{(n)}_t , \bar{\mathbb{X}}^{(n)}_{s,t}\right) \longrightarrow \left(0 , \frac{c}{4}(t-s)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)$$

Example 2

$\mathbb{Z}^2$

X_n: 4 steps of SRW
4 steps of $\square$
and iterate

$$\bar{X}^{(n)} \longrightarrow \frac{1}{\sqrt{2}}B \quad , \quad \bar{\mathbb{X}}^{(n)}_{s,t} \longrightarrow \frac{1}{2}\mathbb{B}^{Str}_{s,t} + \frac{c}{4}(t-s)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

1) Main result to present

Regenerative processes

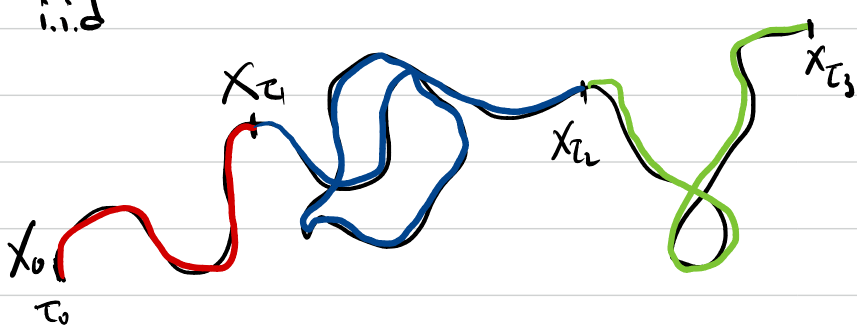
Let (X_n) be a processes on $\mathbb{R}^d$ which is regenarative:

$\exists$ a sequence of random integers

$0 = \tau_0 < \tau_1 < \dots < \infty$ s.t.

$$\left(\tau_{k+1} - \tau_k, \ (X_{\tau_k, \tau_k + n})_{n=0,\dots,\tau_{k+1}-\tau_k}\right)_{k \in \mathbb{N}_0}$$

is i.i.d



<u>Assume</u> $\forall p \in \{0,2\}$, $i = 1, \dots, d$

(M) $\quad 0 < \mathbb{E}\left[(\Xi_0^i)^p T_0 \right] < \infty$

$T_k := \tau_{k+1} - \tau_k$, $\quad \Xi_k^i = \sup_{n=0,\dots,T_k} |X^i_{\tau_k, \tau_k + n}|$

<u>Remarks</u>

① RW $\quad X_n = \sum_{k=1}^{n} \xi_k$, ξ_k i.i.d

is a reg. process: $T_k = 1$, $k = 0, 1, \dots$

(M) becomes $\quad 0 < \mathbb{E}\left|(\xi_k^i)^2\right| < \infty \quad i = 1, \dots, d$

non-degenerate $\qquad$ finite second moment

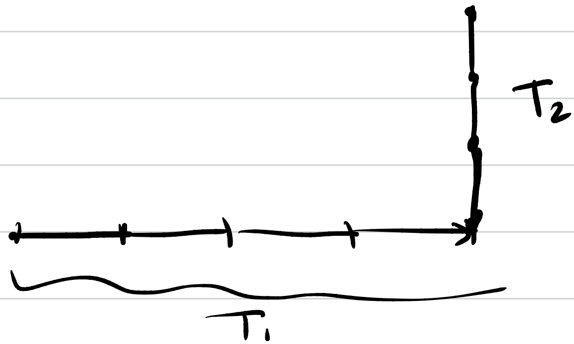
② If T_n are i.i.d with values in $\mathbb{N}$,

$X_n = \sum_{k=1}^{n} \xi_k$ SRW on $\mathbb{Z}^d$

Write $\tau_0 = 0$, $\tau_n = \sum_{k=1}^{n} T_k$.

For $n \in \mathbb{N}_0$ we define

$$Y_{n+1} = Y_n + \xi_{k+1}, \quad \text{w/ } k \text{ is s.t. } \tau_k \le n < \tau_{k+1}$$



$$\left|\Xi^i_k\right| = T_k \underbrace{\left|\xi^i_{k+1}\right|}_{\in\{0,1\}} \quad \left(\xi^i_k \sim \mathrm{Ber}\left(\tfrac{1}{d}\right)\right)$$

$$(M) \Rightarrow 0 < \mathbb{E}\left[T_0^{3}\right] < \infty$$

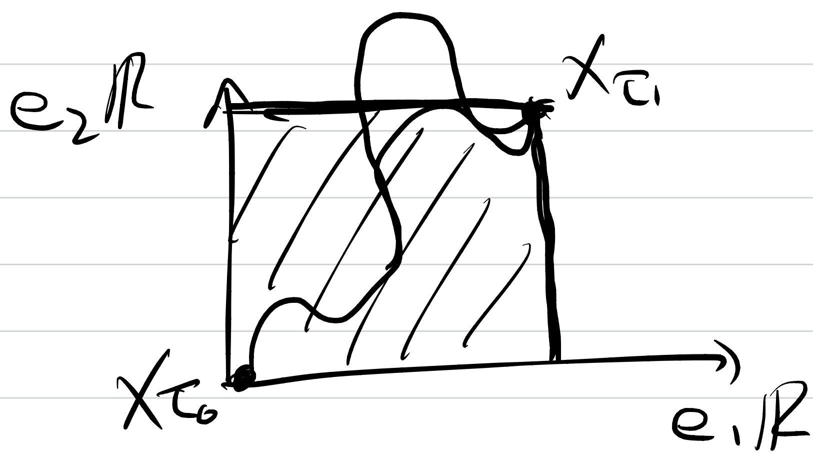
(classic regenerative processes invariance principle)

③ **Known:** (M) yields that

$$\bar{X}^{(n)} \xrightarrow{\text{law}} \Sigma^{1/2} B \quad \text{in } C,$$

where

$$\Sigma^{ij} = \frac{\mathbb{E}\left[X^i_{\tau_1,\tau_2} X^j_{\tau_1,\tau_2}\right]}{\mathbb{E}\left[\tau_2 - \tau_1\right]} = \frac{\mathbb{E}[X^i_{\tau_1} X^j_{\tau_1}]}{\mathbb{E}[\tau_1]}$$

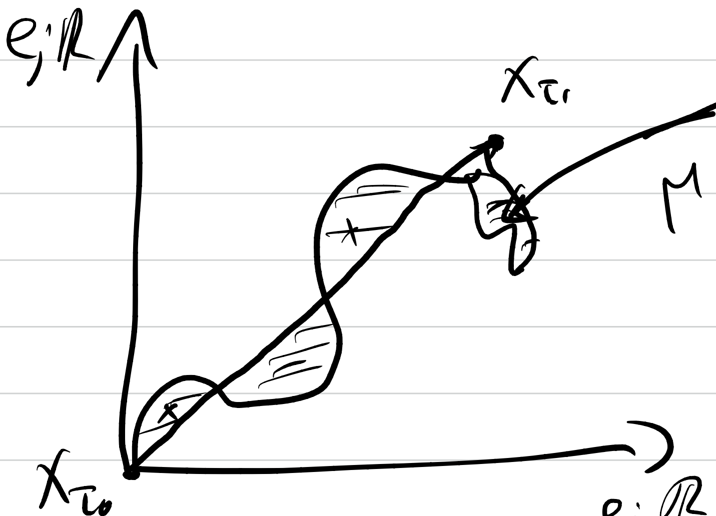


[0 2,1]

<u>Known</u> $X^{(n)}_{s,t} \longrightarrow \int_s^t \beta_{s,u} \circ d\beta_u + \Gamma$

where

$$\Gamma^{ij} = \frac{\mathbb{E}\left[\text{Antisym}\left(\sum_{k=\tau_1+1}^{\tau_2} X_{\tau_1,k-1} \otimes X_{k-1,k}\right)^{ij}\right]}{\mathbb{E}[\tau_2-\tau_1]} = \frac{\mathbb{E}\left[\text{Antisym}\left(\sum_{k=1}^{\tau_1} X_{k-1} \otimes X_{k-1,k}\right)^{ij}\right]}{\mathbb{E}[\tau_1]}$$



$$\Gamma^{ij} = \frac{\mathbb{E}[\ ]}{\mathbb{E}[\tau_1]}$$

Rough invariance principles

Goal: Discuss tools

Settings $F : [0,T]^2_{\le} \longrightarrow \mathbb{R}$, $p \geq 1$

$$\|F\|_{p\text{-var},[0,T]} := \Big(\sup_{\substack{\text{partitions} \\ \pi \text{ of } [0,T]}} \sum_{[s,t] \in \pi} |F_{st}|^p \Big)^{1/p}$$

p-variation norm of F

If $f : [0,T] \to \mathbb{R}^d$ is cadlag

$\|\delta f\|_{p\text{-var},[0,T]}$ is well-defined

$D^p :=$ all F cadlag with finite p-var norm

Morally: p-var is a parameterisation independent version of $\frac{1}{p}$-Hölder

Exercise ① $\|F\|_{p\text{-var},[0,T]} \leq T^{1/p} \|F\|_{\frac{1}{p}}$

② If f is continuous + $\|\delta f\|_{p\text{-var},[0,T]} < \infty$

then $\exists$ reparameterisation σ s.t.

$$\|\delta(f \circ \sigma)\|_{\frac{1}{p}} < \infty$$

<u>Fact</u> B Bm $\Rightarrow \delta B \underset{\text{a.s.}}{\in} D^p([0,T], \mathbb{R}^d)\ \forall p > 2$.

<u>Ref.</u> Friz - Hairer Chap 11.

Cadlag functions. Can define rough paths with

Skorohod topology $D_p([0,T], (\mathbb{R}^d, \mathbb{R}^{d\times d}))$.

$p \in [2,3)$

It's defined as the usual Skorohod topology

with $\|\cdot\|_{\sup}$ replaced by $\|\cdot\|_{p\text{-var},[0,T]}$

see definition ⓐ below.

<u>Reference</u> Friz - Zhang

Lemma Let $(Z^n, \mathbb{Z}^n)$ be a seq. of

cadlag r.p and let $p<3$.

Assume $\exists (Z, \mathbb{Z})$ cadlag p-r.p.

Assume ① $(Z^n_{0,\cdot}, \mathbb{Z}^n_{0,\cdot}) \xrightarrow{\text{law}} (Z_{0,\cdot}, \mathbb{Z}_{0,\cdot})$

in D

② $\left(\|(Z^n, \mathbb{Z}^n)\|_{p\text{-var},[0,T]} \right)$

is tight

$$\left[\|(Z, \mathbb{Z})\|_{p\text{-var},[0,T]} \overset{\Delta}{:=} \|\delta Z\|_{p\text{-var}[0,T]} + \sqrt{\|\mathbb{Z}\|_{p/2\text{-var},[0,T]}} \right]$$

Then $(Z^n, \mathbb{Z}^n) \xrightarrow{\text{law}} (Z, \mathbb{Z})$ in $D_{p'}$

$\forall p<p'<3$.

[Thm 6.7 Friz – Zhang '18]

Theorem (Lépingle–BDG for Itô lifts)

Let $X:[0,T]\to\mathbb{R}^d$ be cadlag local martingale with Ito lift $\mathbb{X}$. Fix $p>2$ and $q\geq 1$.

Then $\exists\, c, C$ s.t for $\mathbf{X} = (X,\mathbb{X})$

$$c\,\mathbb{E}\left[\|\langle X\rangle_T\|^{q/2}\right] \leq \mathbb{E}\left[\|\mathbf{X}\|^q_{p\text{-var},[0,T]}\right] \leq C\,\mathbb{E}\left[\|\langle X\rangle_T\|^{q/2}\right]$$

Theorem (Kurtz – Protter '91)

Consider cadlag (X^n, M^n)

adapted local martingale

Assume $(X^n, M^n) \xrightarrow{\text{mod}} (X, B)$ in D

$I(X,M) :=$ Itô integral of X w.r.t M

(defined in [KP91] as limit in probability of left-point sum w.r.t partitions of mesh size $\to 0$)

"UCV condition"

Assume $\mathbb{E}[\langle M^n \rangle] \le C < \infty \;\forall n$

Then

$$(X^n_\cdot, M^n_\cdot, I_\cdot(X^n, M^n)) \xrightarrow{\text{mod}} (X_\cdot, B_\cdot, I_\cdot(X, B)) \quad \text{in } D,$$

where mod $\in \{\mathbb{P}, \text{Law}\}$.

Martingale invariance principle

[KLO - Fluctuations in Markov Processes]

Thm 2.26

Assume $(M_t)_{t\in[0,\infty)}$ is a d-dimensional cadlag martingale, with good filtration

[i.e satisfies the "usual conditions"]

Assume: M is square integrable with stationary increments

$\forall t \geq 0, m \in \mathbb{N}, s_0 < s_1 < \dots < s_n$

$(M_{s_0, s_1}, \dots, M_{s_{n-1}, s_n})$ and

$(M_{t+s_0, t+s_1}, \dots, M_{t+s_{n-1}, t+s_n})$ have the same distribution.

Let $(\langle M^i, M^j \rangle)_{i,j=1,\dots,d}$ be the predictible covariation process

Assume

$$\frac{\langle M^i, M^j \rangle_t}{t} \xrightarrow[t \to \infty]{\text{in } L^1 \text{ and a.s}} \mathbb{E}[M_1^i M_1^j] = \Sigma^{ij}$$

Then

$$\frac{M_{tn}}{\sqrt{n}} \xrightarrow{\text{law}} \Sigma^{1/2} B \text{ in } D$$

<u>Theorem</u> (Rough Mart. IP)

In the condiens above, $\forall p > 2$

$$(M^{(n)}, \mathbb{M}^{(n)}) \xrightarrow{\text{law}} (\beta, \mathbb{B}) \text{ in } D_p$$

Ito lift

where $\beta = \Sigma^{1/2} B$

$$\mathbb{B}_{s,t} = \int_s^t \beta_{su} \otimes d\beta_u$$

Itô

Proof

① By classical Mart. I.P

$M^{(n)} \xrightarrow{\text{law}} \beta$ in D

② By Kurtz - Protter

$(M^{(n)}, \mathbb{M}^{(n)}_{0,\cdot}) \xrightarrow{\text{law}} (\beta_\cdot, \mathbb{B}_{0,\cdot})$

in D.

③ By p-var Lèpingle-BDG $q=1$, $p>2$

$\mathbb{E}[\|(M^{(n)}, \mathbb{M}^{(n)})\|_{p\text{-var},[0,T]}] \leq C\,\mathbb{E}[\langle M^{(n)}\rangle_T]^{1/2}$

$\lesssim T \| \Sigma \|_{\sup} =: T \max_{i,j \in \{1,\dots,d\}} |\Gamma^{ij}|$

$\Rightarrow$ tightness

$\Rightarrow$ conclude. $\boxed{X}$

Lemma by Friz-Zhang

Back to regenerative processes.

(X_n), τ_n : $(\tau_{k+1}-\tau_k, (X_{\tau_k, \tau_k+n})_{n=0,\dots,\tau_{k+1}-\tau_k})_{k\in\mathbb{N}_0}$ iid

<u>Key identity</u> Set $Z_n := X_{\tau_n}, n \in \mathbb{N}_0$.

<u>Then</u>:

$$\underset{\overline{\mathbb{X}}^{(1)}_{\tau_\ell,\tau_k}}{\overset{}{I^{Str}_{\tau_\ell,\tau_k}(X,X)}} := \underset{\overline{\mathbb{Z}}^{(1)}_{\ell,m}}{I^{Str}_{\ell,m}(Z,Z)} + \sum_{k=\ell}^{m-1} \underset{\mathrm{Anti}(\overline{\mathbb{X}}^{(1)}_{\tau_k,\tau_{k+1}})}{A_{\tau_k,\tau_{k+1}}(X,X)}$$