Lectures on paracontrolled distributions with applications to singular SPDEs

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Homogenisation of a random potential

▷ Consider the linear heat equation with a small random time-independent periodic and smooth (Gaussian) potential V

$$\partial_t U(t,x) = \Delta U(t,x) + \varepsilon^{2-\alpha} V(x) U(t,x), \qquad t \geq 0, x \in \mathbb{T}_\varepsilon^d$$

where $\varepsilon > 0$ is a small parameter, $\alpha < 2$ and $\mathbb{T}_\varepsilon = \mathbb{T}/\varepsilon$,
$\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z}) \sim [0, 2\pi)$.

▷ Introduce macroscopic variables $u_\varepsilon(t,x) = U(t/\varepsilon^2, x/\varepsilon)$ with parabolic rescaling, then

$$\partial_t u_\varepsilon(t,x) = \Delta u_\varepsilon(t,x) + V_\varepsilon(x) u_\varepsilon(t,x), \qquad t \geq 0, x \in \mathbb{T}^d$$

with

$$V_\varepsilon(x) = \varepsilon^{-\alpha} V(x/\varepsilon), \qquad x \in \mathbb{T}^d.$$

Problem: Study the limit $\varepsilon \to 0$ for u_ε.

The random potential

The covariance of the macroscopic noise is

$$\mathbb{E}[V_\varepsilon(x)V_\varepsilon(y)] = \varepsilon^{-2\alpha} C_\varepsilon((x-y)/\varepsilon), \qquad x, y \in \mathbb{T}^d$$

where $C_\varepsilon : \mathbb{T}^d_\varepsilon \to \mathbb{R}$ is a smooth, positive–definite function on $\mathbb{T}^d_\varepsilon$. Assume $\int_{\mathbb{T}^d_\varepsilon} C_\varepsilon(x)dx = 1$.

Take smooth test functions $\varphi, \psi \in \mathscr{S}(\mathbb{T}^d)$ and let $V_\varepsilon(\varphi) = \int_{\mathbb{T}^d} \varphi(x)V_\varepsilon(x)dx$ then

$$\begin{aligned}\mathbb{E}[V_\varepsilon(\varphi)V_\varepsilon(\psi)] &= \varepsilon^{-2\alpha} \int_{\mathbb{T}^d \times \mathbb{T}^d} \varphi(x)\psi(y)C_\varepsilon((x-y)/\varepsilon)dxdy \\ &\sim \varepsilon^{d-2\alpha} \int_{\mathbb{T}^d} \varphi(x)\psi(x)dx \qquad \text{as } \varepsilon \to 0.\end{aligned}$$

Lemma

If $d > 2\alpha$ then $V_\varepsilon \to 0$ in law. If $d = 2\alpha$ then V_ε converges in law to the space white noise ξ on $\mathbb{T}^d$.

White noise on $\mathbb{T}^d$

A family $\{\xi(\varphi)\}_{\varphi \in \mathscr{S}(\mathbb{T}^d)}$ of r.v. such that $\xi(\varphi) \sim \mathscr{N}(0, \|\varphi\|^2_{L^2(\mathbb{T}^d)})$.

Fourier representation

On the covariance C_ε we assume the form

$$C_\varepsilon(x-y) = (\varepsilon/\sqrt{2\pi})^d \sum_{k\in\varepsilon\mathbb{Z}^d} e^{i\langle x-y,k\rangle} R(k) \to_{\varepsilon\to 0} \int_{\mathbb{R}^d} \frac{\mathrm{d}k}{(2\pi)^{d/2}} e^{i\langle x,k\rangle} R(k)$$

where $\mathbb{Z}_0^d = \mathbb{Z}^d \backslash \{0\}$ and $R \in \mathscr{S}(\mathbb{R}^d)$.

There exists a family of centered complex Gaussian random variables $\{g(k)\}_{k\in\mathbb{Z}^d}$ such that $g(k)^* = g(-k)$ and $\mathbb{E}[g(k)g(k')] = \mathbb{I}_{k+k'=0}$ and

$$V_\varepsilon(x) = \frac{\varepsilon^{d/2-\alpha}}{(\sqrt{2\pi})^{d/2}} \sum_{k\in\mathbb{Z}^d} e^{i\langle x,k\rangle} \sqrt{R(\varepsilon k)} g(k)$$

Taking $\alpha = d/2$ we have (as distributions)

$$\xi(x) = (2\pi)^{-d/2} \sqrt{R(0)} \sum_{k\in\mathbb{Z}^d} e^{i\langle x,k\rangle} g(k).$$

Exercise: Show that there exists a version of ξ taking values in $\mathscr{S}'$.

Sobolev regularity

Consider Sobolev spaces H^σ over $\mathbb{T}^d$ with norm

$$\|f\|^2_{H^\sigma(\mathbb{T}^d)} = \sum_{k\in\mathbb{Z}^d} (1+|k|)^{-2\rho} |\mathscr{F}_{\mathbb{T}^d} f(k)|^2.$$

$$\mathbb{E}\|V_\varepsilon\|^2_{H^{-\rho}} = \frac{\varepsilon^{d-2\alpha}}{(\sqrt{2\pi})^d} \sum_{k\in\mathbb{Z}^d} (1+|k|)^{-2\rho} R(\varepsilon k) \sim \varepsilon^{2\rho-2\alpha} \to 0$$

if $\rho > \alpha$ and $d > 2\alpha$. It stays bounded if $d = 2\alpha$ and $\rho > \alpha$. Similarly for $\mathbb{E}\|X_\varepsilon\|^2_{H^{2-\rho}}$.

The white noise ξ belongs to $H^{-\rho}(\mathbb{T}^d)$ for all $\rho < d/2$.

It is possible to show that it is not better: a.s. $\|\xi\|_{H^{-\rho}} = +\infty$ for $\rho \geq d/2$.

Guesswork

As $\varepsilon \to 0$ we guess that $u_\varepsilon \to u$ where

$$\mathscr{L}u = \begin{cases} 0 & \text{if } d > 2\alpha \\ u\xi & \text{if } d = 2\alpha \end{cases}$$

with $\mathscr{L} = \partial_t - \Delta$ the heat operator. This would hold *provided* the solution map

$$\Psi : \eta \mapsto v$$

which sends potentials η to solutions of the *parabolic Anderson model* (PAM)

$$\mathscr{L}v = v\eta$$

is continuous in an appropriate topology in which $(V_\varepsilon)_\varepsilon$ converges.

Littlewood–Paley decomposition

$\varphi : \mathbb{R}^d \to \mathbb{C}$ with polynomial growth defines a *Fourier multiplier*

$$\varphi(\mathrm{D}) : \mathscr{S}' \to \mathscr{S}', \qquad \varphi(\mathrm{D})f = \mathscr{F}^{-1}(\varphi \mathscr{F} f).$$

▷ **Dyadic partition of unity:** $\chi, \rho \in C^\infty(\mathbb{R}^d, \mathbb{R}_+)$ such that

1. $\mathrm{supp}\rho \subseteq \mathscr{B} = \{|x| \leqslant c\}$ and $\mathrm{supp}\rho \subseteq \mathscr{A} = \{a \leqslant |x| \leqslant b\}$
2. $\chi + \sum_{j \geqslant 0} \rho(2^{-j}\cdot) \equiv 1$
3. $\mathrm{supp}(\chi) \cap \mathrm{supp}(\rho(2^{-j}\cdot)) \equiv 0$ for $j \geqslant 1$ and $\mathrm{supp}(\rho(2^{-i}\cdot)) \cap \mathrm{supp}(\rho(2^{-j}\cdot)) \equiv 0$ for all $i, j \geqslant 0$ with $|i - j| \geqslant 1$.

Write $\rho_{-1} = \chi$ and $\rho_j = \rho(2^{-j}\cdot)$ for $j \geqslant 0$.

▷ **Littlewood–Paley blocks:**

$$\Delta_j f = \rho_j(\mathrm{D}) f = \mathscr{F}^{-1}(\rho_j \mathscr{F} f) = K_i * f = \mathscr{F}^{-1}\left(\rho_j \mathscr{F} f\right), \quad j \geqslant -1.$$

where $K_i = (2\pi)^{-d/2} \mathscr{F}^{-1} \rho_i = 2^{id} K(2^i \cdot)$ with $K \in L^1(\mathbb{R}^d)$

Littlewood–Paley decomposition

$$f = \sum_{j \geqslant -1} \Delta_j f = \lim_{j \to \infty} S_j f \qquad \text{for all } f \in \mathscr{S}'.$$

Hölder-Besov spaces

For $\alpha \in \mathbb{R}$, the Hölder-Besov space $\mathscr{C}^\alpha$ is given by $\mathscr{C}^\alpha = B^\alpha_{\infty,\infty}(\mathbb{T}^d, \mathbb{R})$, where

$$B^\alpha_{p,q} = \left\{ f \in \mathscr{S}' : \|f\|_{B^\alpha_{p,q}} = \Big(\sum_{j \geqslant -1} (2^{j\alpha} \|\Delta_j f\|_{L^p})^q \Big)^{1/q} < \infty \right\}.$$

$B^\alpha_{p,q}$ is a Banach space and while the norm $\|\cdot\|_{B^\alpha_{p,q}}$ depends on (χ, ρ), the space $B^\alpha_{p,q}$ does not and any other dyadic partition of unity corresponds to an equivalent norm. Notation: $\|\cdot\|_\alpha = \|\cdot\|_{B^\alpha_{\infty,\infty}}$.

$$\|\Delta_i f\|_{L^\infty} \lesssim 2^{-i\alpha} \|f\|_\alpha$$

By Parseval $B^\alpha_{2,2} = H^\alpha$.

Example

$$\Delta_i \delta_0(x) = (K_i * \delta_0)(x) = K_i(x) = 2^{id} K(2^i x) \Rightarrow \|\Delta_i \delta_0\|_{L^\infty(\mathbb{T}^d)} \simeq 2^{id}$$

so

$$\delta_0 \in \mathscr{C}^{-d}.$$

Tools

Bernstein inequalities

Let $\mathscr{B}$ be a ball and $k \in \mathbb{N}_0$. For any $\lambda \geqslant 1$, $1 \leqslant p \leqslant q \leqslant \infty$, and $f \in L^p$ with $\operatorname{supp}(\mathscr{F}f) \subseteq \lambda\mathscr{B}$ we have

$$\max_{\mu \in \mathbb{N}^d : |\mu| = k} \|\partial^\mu f\|_{L^q} \lesssim_{k,\mathscr{B}} \lambda^{k+d\left(\frac{1}{p}-\frac{1}{q}\right)} \|f\|_{L^p}.$$

Besov embedding

Let $1 \leqslant p_1 \leqslant p_2 \leqslant \infty$ and $1 \leqslant q_1 \leqslant q_2 \leqslant \infty$, and let $\alpha \in \mathbb{R}$. Then $B^{\alpha}_{p_1,q_1}$ is continuously embedded into $B^{\alpha-d(1/p_1-1/p_2)}_{p_2,q_2}$.

An L^2 computation

$$\Delta_i V_\varepsilon(x) = \frac{\varepsilon^{d/2-\alpha}}{(\sqrt{2\pi})^{d/2}} \sum_{k\in\mathbb{Z}^d} e^{i\langle x,k\rangle}\rho_i(k)\sqrt{R(\varepsilon k)}g(k)$$

so

$$\begin{aligned} \mathbb{E}[|\Delta_i V_\varepsilon(x)|^2] &= \varepsilon^d(\sqrt{2\pi})^d\varepsilon^{-2\alpha}\textstyle\sum_{k\in\mathbb{Z}^d}\rho_i(k)^2 e^{i\langle x,k\rangle}R(\varepsilon k) \\ &\lesssim \varepsilon^{d-2\alpha}2^{id}\sup_{k\in\varepsilon 2^i\mathscr{A}} R(k), \end{aligned} \qquad (1)$$

where $\mathscr{A}$ is the annulus in which ρ is supported. Now if $\varepsilon 2^i \leqslant 1$ we have $\mathbb{E}[|\Delta_i V_\varepsilon(x)|^2] \lesssim 2^{id}\varepsilon^{d-2\alpha} = \varepsilon^{\beta-2\alpha}2^{i\beta}$. The assumption $d-2\alpha \geqslant 0$ then implies $\mathbb{E}[|\Delta_i V_\varepsilon(x)|^2] \lesssim 2^{(2\alpha+\kappa)i}\varepsilon^\kappa$ for any $0 \leqslant \kappa \leqslant d-2\alpha$. In the case $\varepsilon 2^i > 1$ we use that $\int_{B(0,1)^c} R(k)\mathrm{d}k < +\infty$ to estimate

$$\varepsilon^d \sum_{k\in\mathbb{Z}^d} R(\varepsilon k) \lesssim \int_{\mathbb{R}^d} R(k)\mathrm{d}k < +\infty,$$

and then $\mathbb{E}[|\Delta_i V_\varepsilon(x)|^2] \lesssim \varepsilon^{-2\alpha} \lesssim 2^{2\alpha i}(\varepsilon 2^i)^\kappa$ for any small $\kappa > 0$.

Assume $d-2\alpha \geqslant 0$. For any $0 \leqslant \kappa \leqslant d-2\alpha$

$$\mathbb{E}[|\Delta_i V_\varepsilon(x)|^2] \lesssim 2^{(2\alpha+\kappa)i}\varepsilon^\kappa.$$

From L^2 to almost sure behavior

▷ Note that $\Delta_i V_\varepsilon(x)$ is a Gaussian r.v. so for any p

$$\mathbb{E}[\|V_\varepsilon\|^p_{B^{-\rho}_{p,p}}] = \sum_i 2^{-ip\rho} \int_{\mathbb{T}^d} dx \mathbb{E}[|\Delta_i V_\varepsilon(x)|^p] = C_p \sum_i 2^{-ip\rho} \int_{\mathbb{T}^d} dx (\mathbb{E}[|\Delta_i V_\varepsilon(x)|^2])^{p/2}$$

$$\lesssim \sum_i 2^{-ip\rho} 2^{p(\alpha+\kappa/2)i} \varepsilon^{p\kappa/2} \lesssim \varepsilon^{p\kappa/2}$$

for all $\rho > \alpha + \kappa/2$.

▷ By Besov embedding $\|V_\varepsilon\|_{B^{-\rho}_{\infty,\infty}} \lesssim \|V_\varepsilon\|_{B^{-\rho+d/p}_{p,p}}$ so

$$\mathbb{E}[\|V_\varepsilon\|^p_{B^{-\rho}_{\infty,\infty}}] \lesssim \mathbb{E}[\|V_\varepsilon\|^p_{B^{-\rho}_{p,p}}] \lesssim \varepsilon^{p\kappa/2}$$

for all $\rho > \alpha + \kappa/2 + d/p$. Note that κ and p are arbitrary.

Theorem

If $d > 2\alpha$ then $V_\varepsilon \to 0$ in $\mathscr{C}^{-\alpha-}$. While if $d = 2\alpha$ then V_ε converges to the space white noise on $\mathbb{T}^d$ in $\mathscr{C}^{-\alpha-}$.

Regularity of the solution map

We are let to the study of the properties of the equation

$$\mathscr{L}v = \eta v$$

with $\eta \in \mathscr{C}^{-\alpha-}$. This stability is easy to establish when $\alpha < 1$ by standard estimates in Besov spaces. We need two ingredients: $(\gamma = 2 - \alpha -)$

1. Schauder estimates in Besov spaces for the parabolic equation $\mathscr{L}f = g$ in the form $\|f\|_\gamma \lesssim \|g\|_{\gamma-2}$
2. Continuity of the product map $(\eta, v) \mapsto v\eta$ in the form $\|v\eta\|_{\gamma-2} \lesssim \|v\|_\gamma \|\eta\|_{\gamma-2}$

$$v \in \mathscr{C}^\gamma \longrightarrow v\eta \in \mathscr{C}^{\gamma-2} \longrightarrow \Gamma(v) = \mathscr{L}^{-1}(v\eta) \in \mathscr{C}^\gamma$$

Schauder estimates

Let Jf such that $\mathscr{L}Jf = f$ and $Jf(0) = 0$ then

$$Jf(t) = \int_0^t e^{\Delta(t-s)} f_s \mathrm{d}s.$$

Consider $C\mathbb{X} = C([0,T];\mathbb{X})$ and norms $\|f\|_{C_T^\sigma \mathbb{X}} = \sup_{0 \leqslant s < t \leqslant T} \frac{\|f(t)-f(s)\|}{|t-s|^\sigma}$. Let $\mathscr{L}_T^\sigma = C_T\mathscr{C}^\sigma \cap C_T^{\sigma/2}L^\infty$ with the norm $\|\cdot\|_{\mathscr{L}_T^\sigma} = \max\{\|\cdot\|_{C_T\mathscr{C}^\sigma}, \|\cdot\|_{C_T^{\sigma/2}L^\infty}\}$.

If $\sigma \in (0,2)$ then

$$\|Jf\|_{\mathscr{L}_T^\sigma} \lesssim (1+T)\|f\|_{C_T\mathscr{C}^{\sigma-2}}$$

$$\|t \mapsto P_t u\|_{\mathscr{L}_T^\sigma} \lesssim \|u\|_\sigma.$$

Product and paraproduct estimates

Deconstruction of a product: $f \in \mathscr{C}^\rho$, $g \in \mathscr{C}^\gamma$

$$fg = \sum_{i,j \geq -1} \Delta_i f \Delta_j g = f \prec g + f \circ g + f \succ g$$

$$f \prec g = g \succ f = \sum_{i < j-1} \Delta_i f \Delta_j g \qquad f \circ g = \sum_{|i-j| \leq 1} \Delta_i f \Delta_j g$$

Paraproduct (Bony, Meyer et al.)

$$f \prec g \in \mathscr{C}^{\min(\gamma+\rho,\gamma)}$$

$$f \circ g \in \mathscr{C}^{\gamma+\rho} \qquad \text{only if } \gamma + \rho > 0$$

Proof. Recall $f \in \mathscr{C}^\rho$, $g \in \mathscr{C}^\gamma$.

$$i \ll j \Rightarrow \operatorname{supp}\mathscr{F}(\Delta_i f \Delta_j g) \subseteq 2^j \mathscr{A} \qquad i \sim j \Rightarrow \operatorname{supp}\mathscr{F}(\Delta_i f \Delta_j g) \subseteq 2^j \mathscr{B}$$

So if $\rho > 0$

$$\Delta_q(f \prec g) = \sum_{j:j\sim q} \sum_{i:i<j-1} \underbrace{\Delta_q(\Delta_i f \Delta_j g)}_{O(2^{-i\rho - j\gamma})} = O(2^{-q\gamma}) \Rightarrow f \prec g \in \mathscr{C}^\gamma,$$

while if $\rho < 0$

$$\Delta_q(f \prec g) = \sum_{j:j\sim q} \sum_{i:i<j-1} \underbrace{\Delta_q(\Delta_i f \Delta_j g)}_{O(2^{-i\rho - j\gamma})} = O(2^{-q(\gamma+\rho)}) \Rightarrow f \prec g \in \mathscr{C}^{\gamma+\rho}.$$

Finally for the resonant term we have

$$\Delta_q(f \circ g) = \sum_{i\sim j \gtrsim q} \Delta_q(\Delta_i f \Delta_j g) = \sum_{i \gtrsim q} O(2^{-j(\rho+\gamma)}) \Rightarrow f \circ g \in \mathscr{C}^{\gamma+\rho}$$

but *only if* the sum converges.

Continuity of PAM for $\gamma > 1$

Assume that $\gamma > 1$. Let

$$\Gamma(v)(t) = P_t v(0) + J(v\eta)(t)$$

and assume that $v(0) \in \mathscr{C}^{\gamma}$. By the product estimate

$$(v, \eta) \in \mathscr{L}^{\gamma} \times \mathscr{C}^{\gamma-2} \to v\eta \in \mathscr{L}^{\gamma-2}$$

if $2\gamma - 2 > 0$. In this case by Schauder estimates $J(u\eta) \in \mathscr{L}^{\gamma}$ so

$$v \in \mathscr{L}^{\gamma} \longrightarrow v\eta \in \mathscr{L}^{\gamma-2} \longrightarrow \Gamma(v) = \mathscr{L}^{-1}(v\eta) \in \mathscr{L}^{2-\alpha}.$$

The map $\Psi : \eta \mapsto v$ is continuous form $\mathscr{C}^{\gamma-2} \to \mathscr{L}^{\gamma}$

If $\gamma \leq 1$ the above argument breaks down since

$$(v, \eta) \in \mathscr{C}^{\gamma} \times \mathscr{C}^{\gamma-2} \not\to v\eta \in \mathscr{C}^{\gamma-2}$$

(it is not continuous).

Enhanching PAM

Let X the solution to

$$\mathscr{L}X = \eta, \qquad X(0,\cdot) = 0$$

and let $v = e^X w$. Then

$$\mathscr{L}v = e^X \mathscr{L}w + e^X w \mathscr{L}X - e^X w|\partial_x X|^2 - e^X \langle \partial_x X, \partial_x w\rangle = v\eta$$

so

$$\mathscr{L}w = |\partial_x X|^2 + \langle \partial_x X, \partial_x w\rangle$$

Take $\eta = V_\varepsilon$ and $\mathscr{L}X_\varepsilon = V_\varepsilon$ then

$$\partial_x X_\varepsilon(t,x) = \int_0^t \int_{\mathbb{T}^d} \partial_x p(t-s, x-y) V_\varepsilon(y) dy ds$$

$$= \frac{\varepsilon^{d/2-\alpha}}{(\sqrt{2\pi})^{d/2}} \sum_{k \in \mathbb{Z}_0^d} \int_0^t ik e^{-|k|^2(t-s)} ds \; e^{i\langle x,k\rangle} \sqrt{R(\varepsilon k)} g(k).$$

Absence of continuity

$$\mathbb{E}[|\partial_x X_\varepsilon(t,x)|^2] = \frac{\varepsilon^{d-2\alpha}}{(\sqrt{2\pi})^d} \sum_{k\in\mathbb{Z}_0^d} \left| \int_0^t ike^{-|k|^2(t-s)} ds \right|^2 R(\varepsilon k)$$

$$= \frac{\varepsilon^{d+2-2\alpha}}{(\sqrt{2\pi})^d} \sum_{k\in\varepsilon\mathbb{Z}_0^d} \frac{|1-e^{-|k/\varepsilon|^2 t}|^2}{|k|^2} R(k) \sim \varepsilon^{2-2\alpha} \int_{\mathbb{R}^d} \frac{R(k)}{|k|^2}$$

If $d > 2$ and $\alpha = 1$ we have $V_\varepsilon, X_\varepsilon \to 0$ but $\mathbb{E}[|\partial_x X_\varepsilon(t,x)|^2] \to \sigma^2 > 0$!
If $d = 2$ and $\alpha = 1$ it even happens that $\mathbb{E}[|\partial_x X_\varepsilon(t,x)|^2] \sim |\log \varepsilon| \to +\infty$.

Note that $\partial_x X_\varepsilon \in C\mathscr{C}^{\gamma-1}$ (uniformly in ε) and by product estimates $X_\varepsilon \mapsto |\partial_x X_\varepsilon|^2$ is continuous only if $\gamma > 1$.

This example show optimality of the condition for the continuity of the product.

Fluctuations of $|\partial_x X_\varepsilon|^2$

Compute

$$\Delta_q(|\partial_x X_\varepsilon|^2)(t,x) = \frac{\varepsilon^{d-2\alpha}}{(2\pi)^{d/2}} \sum_{k_1,k_2\in\mathbb{Z}_0^d} e^{i\langle k_1+k_2,x\rangle}\rho_q(k_1+k_2)G_\varepsilon(t,\varepsilon k_1)G_\varepsilon(t,\varepsilon k_2)g(k_1)g(k_2).$$

where

$$G_\varepsilon(t,k) = i\frac{k}{\varepsilon}\frac{[1-e^{-t|k/\varepsilon|^2}]}{|k/\varepsilon|^2}\sqrt{R(k)}.$$

By Wick's theorem

$$\begin{aligned}\mathrm{Cov}(g(k_1)g(k_2),g(k_1')g(k_2')) &= \mathbb{E}[g(k_1)g(k_1')]\mathbb{E}[g(k_2)g(k_2')] \\ &\quad + \mathbb{E}[g(k_1)g(k_2')]\mathbb{E}[g(k_2)g(k_1')] \\ &= \mathbb{I}_{k_1+k_1'=k_2+k_2'=0} + \mathbb{I}_{k_1+k_2'=k_2+k_1'=0},\end{aligned}$$

which implies

$$\mathrm{Var}[\Delta_q(|\partial_x X_\varepsilon|^2)(t,x)] = \frac{\varepsilon^{2d-4\alpha}}{(\sqrt{2\pi})^{2d}} \sum_{k_1,k_2\in\mathbb{Z}_0^d} (\rho_q(k_1+k_2))^2|G(\varepsilon k_1)|^2|G(\varepsilon k_2)|^2.$$

On one side we have

$$\mathrm{Var}[\Delta_q(|\partial_x X_\varepsilon|^2)(t,x)] \lesssim \varepsilon^{2d+4-4\alpha} \sum_{k_1,k_2\in\varepsilon\mathbb{Z}_0^d} (\rho_q((k_1+k_2)/\varepsilon))^2 \frac{|R(k_1)||R(k_2)|}{|k_1|^2|k_2|^2}.$$

$$\lesssim \varepsilon^{2d+4-4\alpha} \sum_{k_1,k_2\in\varepsilon\mathbb{Z}_0^d} \frac{|R(k_1)||R(k_2)|}{|k_1|^2|k_2|^2} \lesssim \varepsilon^{4-4\alpha} \left(\int \mathrm{d}k \frac{|R(k)|}{|k|^2}\right)^2$$

On the other side in order to satisfy $k_1+k_2\sim\varepsilon 2^q$ we must have $k_2 \lesssim k_1 \sim \varepsilon 2^q$ or $\varepsilon 2^q \lesssim k_1 \sim k_2$. In the first case

$$\varepsilon^{2d+4-4\alpha} \sum_{k_1,k_2\in\varepsilon\mathbb{Z}_0^d} \mathbb{I}_{k_2\lesssim k_1\sim\varepsilon 2^q} \frac{|R(k_1)||R(k_2)|}{|k_1|^2|k_2|^2} \lesssim 2^{q(d-2)}\varepsilon^{2d+2-4\alpha} \sum_{k_2\in\varepsilon\mathbb{Z}_0^d} \mathbb{I}_{k_2\lesssim\varepsilon 2^q} \frac{|R(k_2)|}{|k_2|^2}$$

$$\lesssim (\varepsilon 2^q)^{d-2}\varepsilon^{4-4\alpha}\|R\|_\infty \int \mathrm{d}k \frac{|R(k)|}{|k|^2} \lesssim (\varepsilon 2^q)^{d-2}\varepsilon^{4-4\alpha}\|R\|_\infty \sigma^2$$

since $|R(k_1)|/|k_1|^2 \lesssim \|R\|_\infty/(\varepsilon 2^q)^2$.

If $\varepsilon 2^q \lesssim k_1 \sim k_2$ we similarly have

$$\varepsilon^{2d+4-4\alpha} \sum_{k_1,k_2\in\varepsilon\mathbb{Z}_0^d} \mathbb{I}_{\varepsilon 2^q \lesssim k_1 \sim k_2} \frac{|R(k_1)||R(k_2)|}{|k_1|^2|k_2|^2}$$

$$\lesssim 2^{q(d-2)}\varepsilon^{2d+2-4\alpha}\|R\|_\infty \sum_{k_2\in\varepsilon\mathbb{Z}_0^d} \mathbb{I}_{\varepsilon 2^q\lesssim k_2} \frac{|R(k_2)|}{|k_2|^2} \lesssim (\varepsilon 2^q)^{d-2}\varepsilon^{4-4\alpha}\|R\|_\infty \sigma^2$$

so we can conclude that

$$\mathrm{Var}[\Delta_q(|\partial_x X_\varepsilon|^2)(t,x)] \lesssim \varepsilon^{4-4\alpha}\min(\sigma^4, (\varepsilon 2^q)^{d-2}\|R\|_\infty\sigma^2).$$

Let $c_\varepsilon(t) = \mathbb{E}[|\partial_x X_\varepsilon|^2(t,x)]$ and $|\partial_x X_\varepsilon|^{\diamond 2} = |\partial_x X_\varepsilon|^2 - c_\varepsilon$

By hypercontractivity of Gaussian measures

$$\mathbb{E}[||\partial_x X_\varepsilon|^{\diamond 2}(t,x)|^p] \lesssim_p (\mathbb{E}[||\partial_x X_\varepsilon|^{\diamond 2}(t,x)|^2])^{p/2} \lesssim (\varepsilon^{4-4\alpha}\min(1,(\varepsilon 2^q)^{d-2}))^{p/2}$$

Let $\alpha = 1$ then when $d > 2$, $|\partial_x X_\varepsilon|^{\diamond 2} \to 0$ and $|\partial_x X_\varepsilon|^2 \to c_\varepsilon$ in $C_{[\delta,T]}\mathscr{C}^{0-}$.
and when $d = 2$, $|\partial_x X_\varepsilon|^{\diamond 2} \to |\partial_x X|^{\diamond 2}$ in $C_T\mathscr{C}^{0-}$.

Continuity of the transformed problem

Consider

$$\mathscr{L}w = \theta + \langle \partial_x X, \partial_x w \rangle$$

with $X \in C\mathscr{C}^\gamma$ and $\theta \in C\mathscr{C}^{2\gamma-2}$. This equation can be solved for $w \in C\mathscr{C}^{2\gamma}$

$$(\partial_x X, \partial_x w) \in C\mathscr{C}^{\gamma-1} \times C\mathscr{C}^{2\gamma-1} \mapsto \langle \partial_x X, \partial_x w \rangle \in C\mathscr{C}^{3\gamma-2}$$

is continuous if $3\gamma - 2 > 0$. In this case we have

$$\theta + \langle \partial_x X, \partial_x w \rangle \in C\mathscr{C}^{2\gamma-2} \Rightarrow J(\theta + \langle \partial_x X, \partial_x w \rangle) \in C\mathscr{C}^{2\gamma}$$

If $3\gamma - 2 > 0$ there exists a continuous map

$$\Psi : (X, \theta) \in C\mathscr{C}^\gamma \times C\mathscr{C}^{2\gamma-2} \mapsto w \in C\mathscr{C}^\gamma$$

Lack of continuity, revisited

Setting $w_\varepsilon = \Psi(JV_\varepsilon, |\partial_x JV_\varepsilon|^2)$ and $u_\varepsilon = e^{JV_\varepsilon} w_\varepsilon$ we have that

$$\mathscr{L} u_\varepsilon = u_\varepsilon V_\varepsilon$$

Let $\alpha = 1$ and $d > 2$. When $\varepsilon \to 0$ $JV_\varepsilon \to 0$ in $C\mathscr{C}^\gamma$ and $|\partial_x JV_\varepsilon|^2$ in $C\mathscr{C}^{2\gamma-2}$ which implies

$$w_\varepsilon \to w = \Psi(0, \sigma^2), \qquad u_\varepsilon \to u = w$$

respectively in $C\mathscr{C}^{2\gamma}$ and $C\mathscr{C}^\gamma$.

Now

$$\mathscr{L} u_\varepsilon = u_\varepsilon V_\varepsilon$$

but

$$\mathscr{L} u = \sigma^2 \neq 0.$$

Showing that the limit is not what we expected! Even worse when $d = 2$

since now

$$|\partial_x JV_\varepsilon|^2 \to +\infty + |\partial_x J\xi|^{\diamond 2}$$

A first renormalization

Introduce the renormalized variable

$$\tilde{u}_\varepsilon(t) = e^{-\int_0^t c_\varepsilon(s)ds} u_\varepsilon(t)$$

solving

$$\mathscr{L}\tilde{u}_\varepsilon = V_\varepsilon \tilde{u}_\varepsilon - c_\varepsilon \tilde{u}_\varepsilon$$

Then

$$\mathscr{L}\tilde{w}_\varepsilon = (|\partial_x X_\varepsilon|^2 - c_\varepsilon) + \langle \partial_x X_\varepsilon, \partial_x \tilde{w}_\varepsilon \rangle$$

So now $\tilde{w}_\varepsilon = \Psi(X_\varepsilon, |\partial_x X_\varepsilon|^2 - c_\varepsilon)$ and when $\varepsilon \to 0$ we have

$$\tilde{w}_\varepsilon \to \tilde{w} = \Psi(X, |\partial_x X|^{\diamond 2})$$

In this case the limit is still random. What is the equation satisfied by $\tilde{u} = e^X \tilde{w}$?

Formally

$$\mathscr{L}\tilde{u} = ''\xi\tilde{u} - \infty\tilde{u}''.$$

Both terms in the r.h.s. are not well defined but their sum is.

Paracontrolled analysis

In order to give a meaning to the PDE for v when $\gamma < 1$ we need to understand the properties of the product $v\xi$.

Note that $X\xi$ can be given a well defined meaning by the formula

$$X\xi = X\mathscr{L}X = \mathscr{L}X^2 + |\partial_x X|^2$$

so that

$$X_\varepsilon V_\varepsilon - c_\varepsilon = \mathscr{L}X_\varepsilon^2 + |\partial_x X_\varepsilon|^{\diamond 2}$$

and then by taking limits we have

$$''X\xi - \infty'' = \mathscr{L}X^2 + |\partial_x X|^{\diamond 2}$$

We would like to say that $v = e^X w$ is somewhat as irregular as X (since w is twice as regular) and use this to control $v\xi$ as we were able to control $X\xi$.

A possible rigorous formulation of this "as irregular as" is given by *paracontrolled distributions*. We want to show that there exists a function v^X such that

$$v - v^X \prec X \in C\mathscr{C}^{2\gamma}$$

and that this will help us in the analysis of $v\xi$.

Paralinearization

Lemma

Let $\alpha \in (0,1)$, $\beta \in (0,\alpha]$, and let $F \in C_b^{1+\beta/\alpha}$. There exists a locally bounded map $R_F : \mathscr{C}^\alpha \to \mathscr{C}^{\alpha+\beta}$ such that

$$F(f) = F'(f) \prec f + R_F(f) \tag{2}$$

for all $f \in \mathscr{C}^\alpha$. More precisely, we have

$$\|R_F(f)\|_{\alpha+\beta} \lesssim \|F\|_{C_b^{1+\beta/\alpha}} (1 + \|f\|_\alpha^{1+\beta/\alpha}).$$

If $F \in C_b^{2+\beta/\alpha}$, then R_F is locally Lipschitz continuous:

$$\|R_F(f) - R_F(g)\|_{\alpha+\beta} \lesssim \|F\|_{C_b^{2+\beta/\alpha}} (1 + \|f\|_\alpha + \|g\|_\alpha)^{1+\beta/\alpha} \|f - g\|_\alpha.$$

Proof of paralinearization

The difference $F(f) - F'(f) \prec f$ is given by

$$R_F(f) = F(f) - F'(f) \prec f = \sum_{i \geqslant -1} [\Delta_i F(f) - S_{i-1} F'(f) \Delta_i f] = \sum_{i \geqslant -1} u_i,$$

and every u_i is spectrally supported in a ball $2^i \mathscr{B}$. For $i < 1$, we simply estimate $\|u_i\|_{L^\infty} \lesssim \|F\|_{C_b^1}(1 + \|f\|_\alpha)$. For $i \geqslant 1$

$$\begin{aligned} u_i(x) &= \int K_i(x-y) K_{<i-1}(x-z)[F(f(y)) - F'(f(z))f(y)] \mathrm{d}y\mathrm{d}z \\ &= \int K_i(x-y) K_{<i-1}(x-z)[F(f(y)) - F(f(z)) - F'(f(z))(f(y) - f(z))] \mathrm{d}y\mathrm{d}z, \end{aligned}$$

where $K_i = \mathscr{F}^{-1}\rho_i$, $K_{<i-1} = \sum_{j<i-1} K_j$, and where we used that $\int K_i(y)\mathrm{d}y = \rho_i(0) = 0$ for $i \geqslant 0$ and $\int K_{<i-1}(z)\mathrm{d}z = 1$ for $i \geqslant 1$.

Proof of paralinearization (continued)

Now we can apply a first order Taylor expansion to F and use the β/α–Hölder continuity of F' in combination with the α–Hölder continuity of f, to deduce

$$\begin{aligned}|u_i(x)| &\lesssim \|F\|_{C_b^{1+\beta/\alpha}} \|f\|_\alpha^{1+\beta/\alpha} \int |K_i(x-y)K_{<i-1}(x-z)| \times |z-y|^{\alpha+\beta} \mathrm{d}y\mathrm{d}z \\ &\lesssim \|F\|_{C_b^{1+\beta/\alpha}} \|f\|_\alpha^{1+\beta/\alpha} 2^{-i(\alpha+\beta)}.\end{aligned}$$

The estimate for $R_F(f)$ follows.

The estimate for $R_F(f) - R_F(g)$ is shown in the same way.

□

Commutator lemma

Lemma

Assume that $\alpha, \beta, \gamma \in \mathbb{R}$ *are such that* $\alpha + \beta + \gamma > 0$ *and* $\beta + \gamma \neq 0$. *Then for* $f, g, h \in C^\infty$ *the trilinear operator*

$$C(f,g,h) = ((f \prec g) \circ h) - f(g \circ h)$$

allows for the bound

$$\|C(f,g,h)\|_{\beta+\gamma} \lesssim \|f\|_\alpha \|g\|_\beta \|h\|_\gamma, \tag{3}$$

and can thus be uniquely extended to a bounded trilinear operator from $\mathscr{C}^\alpha \times \mathscr{C}^\beta \times \mathscr{C}^\alpha$ *to* $\mathscr{C}^{\beta+\gamma}$.

Proof of the commutator lemma

Assume $\beta + \gamma < 0$. By definition

$$C(f,g,h) = \sum_{i,j,k,\ell} \Delta_i(\Delta_j f \Delta_k g)\Delta_\ell h(\mathbb{I}_{j<k-1}\mathbb{I}_{|i-\ell|\leqslant 1} - \mathbb{I}_{|k-\ell|\leqslant 1})$$
$$= \sum_{i,j,k,\ell} \Delta_i(\Delta_j f \Delta_k g)\Delta_\ell h(\mathbb{I}_{j<k-1}\mathbb{I}_{|i-\ell|\leqslant 1}\mathbb{I}_{|k-\ell|\leqslant N} - \mathbb{I}_{|k-\ell|\leqslant 1}),$$

where we used that $S_{k-1}f\Delta_k g$ has support in an annulus $2^k\mathscr{A}$, so that $\Delta_i(S_{k-1}f\Delta_k g) \neq 0$ only if $|i-k| \leqslant N-1$ for some fixed $N \in \mathbb{N}$, which in combination with $|i-\ell| \leqslant 1$ yields $|k-\ell| \leqslant N$. Now for fixed k, the term $\sum_\ell \mathbb{I}_{2\leqslant|k-\ell|\leqslant N}\Delta_k g\Delta_\ell h$ is spectrally supported in an annulus $2^k\mathscr{A}$, so that $\sum_{k,\ell} \mathbb{I}_{2\leqslant|k-\ell|\leqslant N}\Delta_k g\Delta_\ell h \in \mathscr{C}^{\beta+\gamma}$ and we may add and subtract $f\sum_{k,\ell} \mathbb{I}_{2\leqslant|k-\ell|\leqslant N}\Delta_k g\Delta_\ell h$ to $C(f,g,h)$ while maintaining the bound (3).

Proof of the commutator lemma (continued)

It remains to treat

$$\sum_{i,j,k,\ell} \Delta_i(\Delta_j f \Delta_k g)\Delta_\ell h \mathbb{I}_{|k-\ell|\leqslant N}(\mathbb{I}_{j<k-1}\mathbb{I}_{|i-\ell|\leqslant 1} - 1)$$
$$= - \sum_{i,j,k,\ell} \Delta_i(\Delta_j f \Delta_k g)\Delta_\ell h \mathbb{I}_{|k-\ell|\leqslant N}(\mathbb{I}_{j\geqslant k-1} + \mathbb{I}_{j<k-1}\mathbb{I}_{|i-\ell|>1}). \quad (4)$$

We estimate both terms on the right hand side separately. For $m \geqslant -1$ we have

$$\Big\| \Delta_m \Big(\sum_{i,j,k,\ell} \Delta_i(\Delta_j f \Delta_k g)\Delta_\ell h \mathbb{I}_{|k-\ell|\leqslant N}\mathbb{I}_{j\geqslant k-1}\Big)\Big\|_{L^\infty}$$
$$\leqslant \sum_{j,k,\ell} \mathbb{I}_{|k-\ell|\leqslant N}\mathbb{I}_{j\geqslant k-1} \|\Delta_m(\Delta_j f \Delta_k g \Delta_\ell h)\|_{L^\infty} \lesssim \sum_{j\gtrsim m}\sum_{k\lesssim j} 2^{-j\alpha}\|f\|_\alpha 2^{-k\beta}\|g\|_\beta 2^{-k\gamma}\|h\|_\gamma$$
$$\lesssim \sum_{j\gtrsim m} 2^{-j(\alpha+\beta+\gamma)}\|f\|_\alpha\|g\|_\beta\|h\|_\gamma \lesssim 2^{-m(\alpha+\beta+\gamma)}\|f\|_\alpha\|g\|_\beta\|h\|_\gamma,$$

using $\beta + \gamma < 0$.

Proof of the commutator lemma (end)

It remains to estimate the second term in (4). For $|i-\ell|>1$ and $i\sim k\sim \ell$, any term of the form $\Delta_i()\Delta_\ell()$ is spectrally supported in an annulus $2^\ell\mathscr{A}$, and therefore

$$
\begin{aligned}
&\Big\|\Delta_m\Big(\sum_{i,j,k,\ell}\Delta_i(\Delta_j f\Delta_k g)\Delta_\ell h\mathbb{I}_{|k-\ell|\leqslant N}\mathbb{I}_{j<k-1}\mathbb{I}_{|i-\ell|>1}\Big)\Big\|_{L^\infty}\\
&\lesssim \sum_{i,j,k,\ell}\mathbb{I}_{j<k-1}\mathbb{I}_{i\sim k\sim\ell\sim m}\|\Delta_i(\Delta_j f\Delta_k g)\Delta_\ell h\|_{L^\infty}\\
&\lesssim \sum_{j\lesssim m}2^{-j\alpha}\|f\|_\alpha 2^{-m\beta}\|g\|_\beta 2^{-m\gamma}\|h\|_\gamma\lesssim 2^{-m(\beta+\gamma)}\|f\|_\alpha\|g\|_\beta\|h\|_\gamma.
\end{aligned}
$$

□

Paracontrolled analysis of $v = e^X w$

By paralinearization we have

$$e^X = e^X \prec X + C\mathscr{C}^{2\gamma}$$

Using the fact that

$$\|f \prec (g \prec h) - (fg) \prec h\|_{\alpha+\beta} \lesssim \|f\|_\alpha \|g\|_\alpha \|h\|_\beta,$$

we have also

$$e^X w = w \prec (e^X \prec X + C\mathscr{C}^{2\gamma}) + e^X \circ w + w \prec e^X = (e^X w) \prec X + C\mathscr{C}^{2\gamma}$$

which means indeed that

$$v - v^X \prec X \in C\mathscr{C}^{2\gamma}$$

with $v^X = v$.

The Good, the Ugly, the Bad

The product $v\xi$ can be decomposed as

$$v\xi = \underbrace{v \prec \xi}_{\text{The Bad, } \in C\mathscr{C}^{\gamma-2}} + \underbrace{v \circ \xi}_{\text{The Ugly}} + \underbrace{v \succ \xi}_{\text{The Good, } \in C\mathscr{C}^{2\gamma-2}} .$$

The real problem is given by the resonant term $v \circ \xi$. Using $v^\sharp = v - v^X \prec X \in C\mathscr{C}^{2\gamma}$ we have

$$v \circ \xi = (v^X \prec X) \circ \xi + \underbrace{v^\sharp \circ \xi}_{C\mathscr{C}^{3\gamma-2}}$$

By the commutator lemma:

$$v \circ \xi = v^X (X \circ \xi) + v^\sharp \circ \xi + C\mathscr{C}^{2\gamma-2}$$

So

$$v\xi = \Theta(v^X, v^\sharp, \xi, X \circ \xi) = v \prec \xi + v^X (X \circ \xi) + C\mathscr{C}^{2\gamma-2}$$

where the function Θ is continuous.

Structure of solution and paracontrolled distributions

▷ So in the limit $\varepsilon \to 0$ we have

$$\tilde{u}_\varepsilon V_\varepsilon - c_\varepsilon \tilde{u}_\varepsilon = \tilde{u}_\varepsilon \prec V_\varepsilon + \tilde{u}_\varepsilon (X_\varepsilon \circ V_\varepsilon - c_\varepsilon) + C(\tilde{u}_\varepsilon, X_\varepsilon, V_\varepsilon) + \tilde{u}_\varepsilon^\sharp \circ V_\varepsilon + \tilde{u}_\varepsilon \succ V_\varepsilon$$

$$\to \tilde{u} \prec \xi + \tilde{u}(X \diamond \xi) + C(\tilde{u}, X, \xi) + \tilde{u}^\sharp \circ \xi + \tilde{u} \succ \xi$$

$$=: \tilde{u} \diamond \xi = \Phi(\tilde{u}, \tilde{u}^\sharp, X, X \diamond \xi)$$

where $X \diamond \xi := \lim_{\varepsilon \to 0} (X_\varepsilon \circ V_\varepsilon - c_\varepsilon)$.

▷ **Question:** What is the equation satisfied by $\tilde{u} = \lim_{\varepsilon \to 0} \tilde{u}_\varepsilon$?

Indeed

$$\mathscr{L} \tilde{u} = "\tilde{u}\xi - \infty \tilde{u}" = \tilde{u} \diamond \xi = \Phi(\tilde{u}, \tilde{u}^\sharp, X, X \diamond \xi).$$

Where the r.h.s. is well defined since $\tilde{u}$ is **paracontrolled** by X.

Paracontrolled distributions

Paracontrolled distributions

We say $y \in \mathscr{D}_x^\rho$ if $x \in \mathscr{C}^\gamma$

$$y = y^x \prec x + y^\sharp$$

with $y^x \in \mathscr{C}^\rho$ and $y^\sharp \in \mathscr{C}^{\gamma+\rho}$.

▷ **Paralinearization.** Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a sufficiently smooth function and $x \in \mathscr{C}^\gamma$, $\gamma > 0$. Then

$$\varphi(x) = \varphi'(x) \prec x + \mathscr{C}^{2\gamma}$$

▷ Another commutator: $f, g \in \mathscr{C}^\rho$, $x \in \mathscr{C}^\gamma$

$$f \prec (g \prec h) = (fg) \prec h + \mathscr{C}^{\rho+\gamma}$$

▷ **Stability.** $(\rho \leq \gamma)$

$$\varphi(y) = (\varphi'(y) y^x) \prec x + \mathscr{C}^{\rho+\gamma}$$

so we can take $\varphi(y)^x = \varphi'(y) y^x$.

Solution theory for general signals

Goal: Show that $\Psi : \eta \mapsto u$ factorizes as

$$\eta \xrightarrow{\mathbb{X}} \mathbb{X}(\eta) = (\eta, J\eta \circ \eta) \xrightarrow{\Phi} u$$

$\triangleright$ *Analytic step:* show that when $\gamma \in (2/3, 1)$:

$$\Phi : \mathcal{X} \to \mathscr{C}^\gamma$$

is continuous. $\mathcal{X} = \overline{\mathrm{Im}\mathbb{X}} \subseteq \mathscr{C}^{\gamma-2} \times \mathscr{C}^{2\gamma-2}$ is the space of *enhanced signals* (or rough paths, or models).

But in general $\mathbb{X}$ is **not** a continuous map $\mathscr{C}^{\gamma-2} \to \mathscr{C}^{\gamma-2} \times \mathscr{C}^{2\gamma-2}$.

$\triangleright$ *Probabilistic step:* prove that there exists a "reasonable definition" of $\mathbb{X}(\xi)$ when ξ is a white noise. $\mathbb{X}(\xi)$ is an explicit polynomial in ξ so direct computations are possible.

Tools: Besov embeddings $L^p(\Omega; \mathscr{C}^\theta) \to L^p(\Omega; B^{\theta'}_{p,p}) \simeq B^{\theta'}_{p,p}(L^p(\Omega))$, Gaussian hypercontractivity $L^p(\Omega) \to L^2(\Omega)$, explicit L^2 computations.

Paracontrolled gPAM (I) - the r.h.s.

$u : \mathbb{R}_+ \times \mathbb{T}^2 \to \mathbb{R}$, $\xi \in \mathscr{C}^{\gamma-2}$, $\gamma = 1-$. We want to solve (have uniform bounds for)

$$\mathscr{L}u = F(u)\xi = F(u) \prec \xi + F(u) \circ \xi + F(u) \succ \xi.$$

▷ Paracontrolled ansatz. Take $\mathscr{L}X = \xi$, $X \in \mathscr{C}^\gamma$ and assume that $u \in \mathscr{D}^\gamma_X$:

$$u = u^X \prec X + u^\sharp$$

with $u^\sharp \in \mathscr{C}^{2\gamma}$ and $u^X \in \mathscr{C}^\gamma$.

▷ Paralinearization:

$$F(u) = F'(u) \prec u + \mathscr{C}^{2\gamma} = (F'(u)u^X) \prec X + \mathscr{C}^{2\gamma}$$

▷ Commutator lemma:

$$F(u) \circ \xi = ((F'(u)u^X) \prec X) \circ \xi + \mathscr{C}^{2\gamma} \circ \xi$$

$$= \underbrace{(F'(u)u^X)(X \circ \xi)}_{\in \mathscr{C}^{2\gamma-2}} + \underbrace{C(F'(u)u^X, X, \xi) + \mathscr{C}^{2\gamma} \circ \xi}_{\in \mathscr{C}^{3\gamma-2}}$$

if we *assume* that $(X \circ \xi) \in \mathscr{C}^{2\gamma-2}$.

Paracontrolled gPAM (II) - the l.h.s.

So if u is paracontrolled by X:

$$u = u^X \prec X + u^\sharp$$

and if $X \circ \xi \in \mathscr{C}^{2\gamma-2}$ we have a control on the r.h.s. of the equation:

$$F(u)\xi = \underline{F(u) \prec \xi} + F'(u)u^X(X \circ \xi) + \mathscr{C}^{3\gamma-2}$$

What about the l.h.s.?

$$\mathscr{L}u = \mathscr{L}u^X \prec X + \underline{u^X \prec \xi} + \mathscr{L}u^\sharp - \partial_x u^X \prec \partial_x X$$

so letting $u^X = F(u)$ we have

$$\mathscr{L}u^\sharp = -\mathscr{L}F(u) \prec X + F'(u)F(u)(X \circ \xi) + \mathscr{C}^{2\gamma-2}$$

Paracontrolled gPAM (III) - the paracontrolled fixed point.

The PDE

$$\mathscr{L}u = F(u)\xi$$

is equivalent to the system

$$\begin{aligned}
\partial_t X &= \xi \\
\partial_t u^\sharp &= (F'(u)F(u))(X \circ \xi) - \underbrace{\mathscr{L}f(u) \prec X}_{''\in''\mathscr{C}^{2\gamma-2}} + \underbrace{R(f,u,X,\xi) \circ \xi}_{\in\mathscr{C}^{3\gamma-2}} \\
u &= F(u) \prec X + u^\sharp
\end{aligned}$$

▷ The system can be solved by fixed point (for small time) in the space $\mathscr{D}_X^\gamma$ if we assume that

$$X \in \mathscr{C}^\gamma, \qquad (X \circ \xi) \in \mathscr{C}^{2\gamma-2}.$$

Structure of the paracontrolled solution

▷ When ξ smooth, the solution to

$$\partial_t u = F(u)\xi, \qquad u(0) = u_0$$

is given by $u = \Phi(u_0, \xi, X \circ \xi)$ where

$$\Phi : \mathbb{R}^d \times \mathscr{C}^{\gamma-2} \times \mathscr{C}^{2\gamma-2} \to \mathscr{C}^{\gamma}$$

is continuous for any $\gamma > 2/3$ and $z = \Phi(u_0, \xi, \varphi)$ is given by

$$\begin{cases} z = F(z) \prec X + z^\sharp \\ \partial_t z^\sharp = (F'(z)F(z))\varphi - \underbrace{\mathscr{L}F(z) \prec X}_{''\in''\mathscr{C}^{2\gamma-2}} + \underbrace{R(F, z, X, \xi) \circ \xi}_{\in \mathscr{C}^{3\gamma-2}} \end{cases}$$

▷ If $(\xi^n, X^n \circ \xi^n) \to (\xi, \eta)$ in $\mathscr{C}^{\gamma-2} \times \mathscr{C}^{2\gamma-2}$ and

$$\partial_t u^n = f(u^n)\xi^n, \qquad u(0) = u_0$$

then $u^n \to u = \Phi(u_0, \xi, \eta)$.

Relaxed form of the PDE

▷ Note that in general we can have $\xi^{1,n} \to \xi$, $\xi^{2,n} \to \xi$ and

$$\lim_n X^{1,n} \circ \xi^{1,n} \neq \lim_n X^{2,n} \circ \xi^{2,n}$$

▷ Take ξ^n, ξ smooth but $\xi^n \to \xi$ in $\mathscr{C}^{\gamma-2}$. It can happen that

$$\lim_n X^n \circ \xi^n = X \circ \xi + \varphi \in \mathscr{C}^{2\gamma-1}$$

In this case $u^n \to u$ and $u = \Phi(\xi, X \circ \xi + \varphi)$ solves the equation

$$\mathscr{L} u = F(u)\xi + F'(u)F(u)\varphi.$$

The limit procedure generates correction terms to the equation.

The original equation **relaxes** to another form in which additional terms are generated.

"Ito" form of the PDE

In the smooth setting $u = \Phi(\xi, X \circ \xi + \varphi)$ solves

$$\mathscr{L} u = F(u)\xi + F'(u)F(u)\varphi.$$

If we choose $\varphi = -X \circ \xi$ then

$$v = \Phi(\xi, X \circ \xi + \varphi) = \Phi(\xi, 0)$$

solves

$$\mathscr{L} v = F(v)\xi - F'(v)F(v)X \circ \xi$$

and has the particular property of being a continuous map of $\xi \in \mathscr{C}^{\gamma-2}$ alone.

The renormalization problem

If ξ is the space white noise we have

$$\xi \in \mathscr{C}^{-1-}, \qquad X \in C([0,T];\mathscr{C}^{1-})$$

and

$$X \circ \xi = X \circ \mathcal{L}X = \frac{1}{2}\mathcal{L}(X \circ X) + \frac{1}{2}(DX \circ DX)$$
$$= \frac{1}{2}\mathcal{L}(X \circ X) - (DX \prec DX) + \frac{1}{2}(DX)^2$$

But now

$$\frac{1}{2}(DX)^2 = c + C\mathscr{C}^{0-}$$

with $c = +\infty$!.

No obvious definition of $X \circ \xi$ can be given. But there exists c_ε such that

$$X_\varepsilon \circ \xi_\varepsilon - c_\varepsilon \to "X \diamond \xi" \qquad \text{in } C\mathscr{C}^{0-}.$$

The renormalized gPAM

To cure the problem we add a suitable counterterm to the equation

$$\mathcal{L}u = f(u) \diamond \xi = f(u)\xi - c(f'(u)f(u))$$

this defines a new product, denoted by $\diamond$. Now

$$f(u) \circ \xi - c(f'(u)f(u)) = (f'(u)f(u))(X \circ \xi - c) + C(f'(u)f(u), X, \xi) + R(f, u, X) \circ \xi$$

▷ The renormalized gPAM is equivalent to the equation

$$\begin{aligned}\mathcal{L}u^\sharp = -\mathcal{L}f(u) \prec X + \mathrm{D}f(u) \prec \mathrm{D}X + (f'(u)f(u))(X \circ \xi - c) \\ +C(f'(u)f(u), X, \xi) + R(f, u, X) \circ \xi\end{aligned}$$

together with $u = f(u) \prec X + u^\sharp$ and where

$$X \in \mathscr{C}^{1-}, \qquad X \diamond \xi = (X \circ \xi - c) \in \mathscr{C}^{0-}, \quad u^\sharp \in \mathscr{C}^{2-}.$$

Finally a theorem

Theorem

Let $d = 2$, $\alpha = 1$, $\gamma = 1-$ and small $T > 0$. There exist constants c_ε such that letting u_ε the solution to

$$\mathscr{L} u_\varepsilon = V_\varepsilon F(u_\varepsilon) - c_\varepsilon F'(u_\varepsilon)$$

then $u_\varepsilon \to u$ in $\mathscr{C}^\gamma$ as $\varepsilon \to 0$ and $u \in \mathscr{D}_X^{2\gamma}$ is the unique weak solution in $\mathscr{D}_X^{2\gamma}$ to the equation

$$\mathscr{L} u = \xi \diamond F(u) = F(u) \prec \xi + F'(u)(X \diamond \xi) + G(u^X, u^\sharp, X)$$

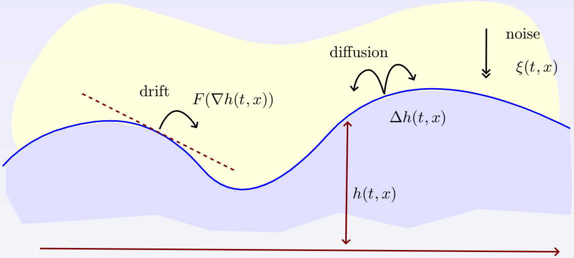
where

$$\xi = \lim_{\varepsilon \to 0} V_\varepsilon, \qquad X \diamond \xi = \lim_{\varepsilon \to 0} X_\varepsilon \circ V_\varepsilon - c_\varepsilon$$

in $\mathscr{C}^{\gamma-2}$ and $\mathscr{C}^{2\gamma-2}$ resp. and ξ has the law of the white noise on $\mathbb{T}^2$.

The KPZ equation

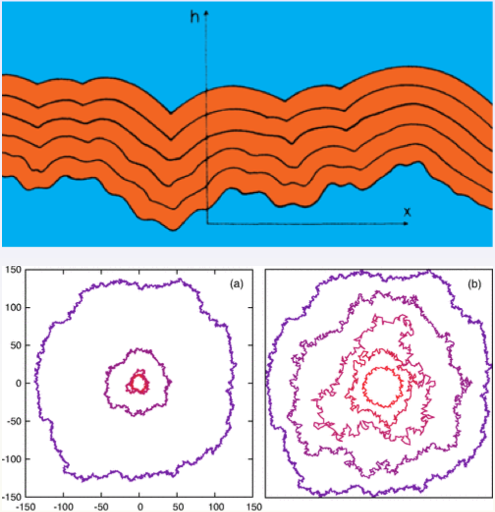
Fluctuations of a growing interface



A model for random interface growth (think e.g. expansion of colony of bacteria): $h\colon \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$,

$$\partial_t h(t,x) = \underbrace{\kappa \Delta h(t,x)}_{\text{relaxation}} + \underbrace{F(\partial_x h(t,x))}_{\text{slope-dependent growth}} + \underbrace{\eta(t,x)}_{\text{noise with microscopic correlations}}$$

Fluctuations of a growing interface



The Kardar–Parisi–Zhang equation

- Kardar–Parisi–Zhang '84: slope-dependent growth given by $F(\partial_x h)$, in a certain scaling regime of small gradients:

$$F(\partial_x h) = F(0) + F'(0)\partial_x h + F''(0)(\partial_x h)^2 + \dots$$

- KPZ equation is the universal model for random interface growth

$$\partial_t h(t,x) = \underbrace{\kappa \Delta h(t,x)}_{\text{relaxation}} + \underbrace{\lambda[(\partial_x h(t,x))^2 - \infty]}_{\text{renormalized growth}} + \underbrace{\xi(t,x)}_{\text{space-time white noise}}$$

- This derivation is highly problematic since $\partial_x h$ is a distribution. But: Hairer, Quastel (2014, unpublished) justify it rigorously via scaling of smooth models and small gradients.
- KPZ equation is suspected to be universal scaling limit for random interface growth models, random polymers, and many particle systems;
- contrary to Brownian setting: KPZ has fluctuations of order $t^{1/3}$; large time limit distribution of $t^{-1/3}h(t, t^{2/3}x)$ is expected to be universal in a sense comparable only to the Gaussian distribution.

KPZ and its siblings:

- KPZ equation:

$$\mathscr{L}h(t,x) = "(\partial_x h(t,x))^2 - \infty" + \xi(t,x);$$

$h\colon \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, $\mathscr{L} = \partial_t - \Delta$ heat operator, ξ space-time white noise;

- Burgers equation:

$$\mathscr{L}u(t,x) = "\partial_x(u(t,x)^2)" + \partial_x \xi(t,x);$$

solution is (formally) given by derivative of the KPZ equation: $u = \partial_x h$;

- solution to KPZ (formally) given by Cole-Hopf transform of the stochastic heat equation: $h = \log w$, where w solves

$$\mathscr{L}w(t,x) = "w(t,x) \diamond \xi(t,x)".$$

- All three are universal objects, that are expected to be scaling limits of a wide range of particle systems.

Stochastic Burgers equation

Take $u = Dh$

$$\mathcal{L}u = D\xi + Du^2$$

to obtain the stochastic Burgers equation (SBE) with additive noise.

▷ **Invariant measure:** Formally the SBE leaves invariant the space white noise: if u_0 has a Gaussian distribution with covariance $\mathbb{E}[u_0(x)u_0(y)] = \delta(x-y)$ then for all $t \geqslant 0$ the random function $u(t,\cdot)$ has a Gaussian law with the same covariance.

▷ **First order approximation:** Let $X(t,x)$ be the solution of the linear equation

$$\partial_t X(t,x) = \partial_x^2 X(t,x) + \partial_x \xi(t,x), \qquad x \in \mathbb{T}, t \geqslant 0$$

X is a stationary Gaussian process with covariance

$$\mathbb{E}[X(t,x)X(s,y)] = p_{|t-s|}(x-y).$$

Almost surely $X(t,\cdot) \in \mathscr{C}^\gamma$ for any $\gamma < -1/2$ and any $t \in \mathbb{R}$. For any $t \in \mathbb{R}$ $X(t,\cdot)$ has the law of the white noise over $\mathbb{T}$.

Expansion for the SBE

Recall the SBE:

$$\mathscr{L}u = Du^2 + \xi$$

▷ Let $u = X + u_1$ then

$$\mathscr{L}u_1 = \partial_x(u_1 + X)^2 = \underbrace{\partial_x X^2}_{-2-} + 2\partial_x(u_1 X) + \partial_x u_1^2$$

▷ Let $X^{\vee}$ be the solution to

$$\mathscr{L}X^{\vee} = \partial_x X^2 \qquad \Rightarrow \qquad X^{\vee} \in \mathscr{C}^{0-}$$

and decompose further $u_1 = X^{\vee} + u_2$. Then

$$\mathscr{L}u_2 = \underbrace{2\partial_x(X^{\vee}X)}_{-3/2-} + 2\partial_x(u_2 X) + \underbrace{\partial_x(X^{\vee}X^{\vee})}_{-1-} + 2\partial_x(u_2 X^{\vee}) + \partial_x(u_2)^2$$

▷ Define $\mathscr{L}X^{\vee\!\!\vee} = 2\partial_x(X^{\vee}X)$ and $u_2 = X^{\vee\!\!\vee} + u_3$ then $X^{\vee\!\!\vee} \in \mathscr{C}^{1/2-}$

$$\mathscr{L}u_3 = \underbrace{2\partial_x(u_3 X)}_{-3/2-} + \underbrace{2\partial_x(X^{\vee\!\!\vee}X)}_{-3/2-} + \underbrace{\partial_x(X^{\vee}X^{\vee})}_{-1-} + 2\partial_x(u_2 X^{\vee}) + \partial_x(u_2)^2$$

Expansion /II

▷ The partial expansion for the solution reads

$$u = X + X^{\vee} + 2X^{\Psi} + U$$

$$\mathscr{L}U = 2\partial_x(UX) + 2\partial_x(X^{\Psi}X) + \partial_x(X^{\vee}X^{\vee}) + 2\partial_x((2X^{\Psi} + U)X^{\vee}) + \partial_x(2X^{\Psi} + U)^2$$

$$= {\color{red}2\partial_x(UX)} + \mathscr{L}(2X^{\Psi} + X^{\vee\vee}) + 2\partial_x((2X^{\Psi} + U)X^{\vee}) + \partial_x(2X^{\Psi} + U)^2$$

and the regularities for the driving terms

X	$X^{\vee}$	X^{Ψ}	$X^{\Psi'}$	$X^{\vee\vee}$
$-1/2-$	$0-$	$1/2-$	$1/2-$	$1-$

We can assume $U \in \mathscr{C}^{1/2-}$ so that the terms

$$2\partial_x((2X^{\Psi} + U)X^{\vee}) + \partial_x(2X^{\Psi} + U)^2$$

are well defined.

The remaining problem is to deal with ${\color{red}2\partial_x(UX)}$.

Paracontrolled ansatz for SBE

▷ Make the following ansatz $U = U' \prec Q + U^\sharp$. Then

$$\mathscr{L}U = \mathscr{L}U' \prec Q + U' \prec \mathscr{L}Q - \partial_x U' \prec \partial_x Q + LU^\sharp$$

while

$$\mathscr{L}U = 2\partial_x(UX) + \underbrace{\mathscr{L}(2X^{\Yleft} + X^{\vee\vee}) + 2\partial_x((2X^{\Yup} + U)X^{\vee}) + \partial_x(2X^{\Yup} + U)^2}_{R(U)}$$

$$= 2\partial_x(U \prec X) + 2\partial_x(U \circ X) + 2\partial_x(U \succ X) + R(U)$$

$$= 2(U \prec \partial_x X) + 2(\partial_x U \prec X) + 2\partial_x(U \circ X) + 2\partial_x(U \succ X) + R(U)$$

so we can set $U' = 2U$ and $\mathscr{L}Q = \partial_x X$ and get the equation

$$\mathscr{L}U^\sharp = -\mathscr{L}U' \prec Q + \partial_x U' \prec \partial_x Q + 2(\partial_x U \prec X) + 2\partial_x(U \circ X) + 2\partial_x(U \succ X) + R(U)$$

▷ Observe that $Q, U, U' \in \mathscr{C}^{1/2-}$ and we can assume that $U^\sharp \in \mathscr{C}^{1-}$.

Commutator

▷ The difficulty is now concentrated in the resonant term $U \circ X$ which is not well defined.

▷ The paracontrolled ansatz and the commutation lemma give

$$U \circ X = (2U \prec Q) \circ X + U^\sharp \circ X = 2U(Q \circ X) + \underbrace{C(2U, Q, X)}_{1/2-} + \underbrace{U^\sharp \circ X}_{1/2-}$$

▷ A stochastic estimate shows that $Q \circ X \in \mathscr{C}^{0-}$

Paracontrolled solution to SBE

▷ The final system reads

$$u = X + X^{\vee} + 2X^{\Yleft} + U$$

$$U = U' \prec Q + U^{\sharp}, \qquad U' = 2X^{\Yleft} + 2U$$

$$\mathscr{L}U^{\sharp} = 4\partial_x(U\underline{(Q \circ X)}) + 4\partial_x C(U,Q,X) + 2\partial_x(U^{\sharp} \circ X) - 2\mathscr{L}U \prec Q$$
$$+2\partial_x U \prec \partial_x Q + 2(\partial_x U \prec X) + 2\partial_x(U \succ X) + R(U)$$

▷ This equation has a (local in time) solution $U = \Phi(\mathbb{X}(\xi))$ which is a continuous function of the data $\mathbb{X}(\xi)$ given by a collection of multilinear functions of ξ:

$$\mathbb{X}(\xi) = (X, X^{\vee}, X^{\Yleft}, X^{\Yleft\!\Yleft}, X^{\vee\!\vee}, X \circ Q)$$

Burgers equation and paracontrolled distributions

$$\mathscr{L} u(t,x) = \partial_x u^2(t,x) + \partial_x \xi(t,x), \qquad u(0) = u_0.$$

Paracontrolled Ansatz

$u \in \mathscr{P}_{\text{rbe}}$ if $u = X + X^{\vee} + 2X^{\curlyvee} + u^Q$ with

$$u^Q = u' \prec Q + u^\sharp.$$

- Paracontrolled structure: Can define u^2 continuously as long as $(Q \circ X) \in C([0,T], \mathscr{C}^{0-})$ is given (together with tree data $X, X^{\vee}, X^{\curlyvee}, X^{\curlyvee\!\!\vee}, X^{\vee\vee}$).
- Obtain local existence and uniqueness of paracontrolled solutions. Solution depends pathwise continuously on extended data $\mathbb{X}(\xi) = (\xi, X, X^{\vee}, X^{\curlyvee}, X^{\curlyvee\!\!\vee}, X^{\vee\vee}, Q \circ X)$.

KPZ equation

KPZ equation:

$$\mathscr{L}h(t,x) = (\partial_x h(t,x))^2 + \xi(t,x), \qquad h(0) = h_0.$$

Expect $h(t) \in \mathscr{C}^{1/2-}$, so $\partial_x h(t) \in \mathscr{C}^{-1/2-}$ and $(\partial_x h(t))^2$ not defined. But: expand

$$u = Y + Y^{\vee} + 2Y^{\vee\!\!\!\!\vee} + h^P,$$

where $\mathscr{L}Y = \xi$, $\mathscr{L}Y^{\vee} = \partial_x Y \partial_x Y, \ldots$ In general: $\partial_x Y^\tau = X^\tau$. Make paracontrolled ansatz for h^P:

$$h^P = \pi_<(h', P) + h^\sharp$$

with $h' \in C([0,T], \mathscr{C}^{1/2-})$, $h^\sharp \in C([0,T], \mathscr{C}^{2-})$, $\mathscr{L}P = X$. Write $h \in \mathscr{P}_{\mathrm{kpz}}$.

Can define $(\partial_x h(t))^2$ for $h \in \mathscr{P}_{\mathrm{kpz}}$ and obtain local existence and uniqueness of solutions.

KPZ and Burgers equation

$h \in \mathscr{P}_{\text{kpz}}$ if

$$h = \Upsilon + \Upsilon^{\vee} + 2\Upsilon^{\vee_{\vee}} + h^P, \qquad h^P = h' \prec P + h^{\sharp}.$$

$u \in \mathscr{P}_{\text{rbe}}$ if

$$u = X + X^{\vee} + 2X^{\vee_{\vee}} + u^Q, \qquad u^Q = u' \prec Q + u^{\sharp}.$$

- If $h \in \mathscr{P}_{\text{kpz}}$, then $\partial_x h \in \mathscr{P}_{\text{rbe}}$.
- If h solves KPZ equation, then $u = \partial_x h$ solves Burgers equation with initial condition $u(0) = \partial_x h_0$.
- If $u \in \mathscr{P}_{\text{rbe}}$, then any solution h of $\mathscr{L}h = u^2 + \xi$ is in $\mathscr{P}_{\text{kpz}}$.
- If u solves Burgers equation with initial condition $u(0) = \partial_x h_0$, and h solves $\mathscr{L}h = u^2 + \xi$ with initial condition $h(0) = h_0$, then h solves KPZ equation.

KPZ and heat equation

Heat equation:

$$\mathscr{L} w(t,x) = w(t,x) \diamond \xi(t,x) = w(t,x)\xi(t,x) - w(t,x)\cdot\infty, \quad w(0) = w_0.$$

Paracontrolled ansatz: $w \in \mathscr{P}_{\mathrm{rhe}}$ if

$$w = e^{Y + Y^{\vee} + 2Y^{\curlyvee}} w^P, \qquad w^P = \pi_<(w', P) + w^\sharp$$

(comes from Cole-Hopf transform).

- Slightly cheat to make sense of product $w \diamond \xi$ for $w \in \mathscr{P}_{\mathrm{rhe}}$:

$$w \diamond \xi = \mathscr{L} w - e^{Y + Y^{\vee} + 2Y^{\curlyvee}} \Big[\mathscr{L} w^P - [\mathscr{L}(Y^{\vee} + Y^{\curlyvee}) + (\partial_x(Y + Y^{\vee} + 2Y^{\curlyvee}))^2] w^P\Big] \\ + 2e^{Y + Y^{\vee} + 2Y^{\curlyvee}} \partial_x(Y + Y^{\vee} + 2Y^{\curlyvee}) \partial_x w^P;$$

  (agrees with renormalized pointwise product $w \diamond \xi$ in smooth case and with Itô integral in white noise case, continuous in extended data).
- Obtain global existence and uniqueness of solutions.
- One-to-one correspondence between $\mathscr{P}_{\mathrm{kpz}}$ and strictly positive elements of $\mathscr{P}_{\mathrm{rhe}}$.
- Any solution of KPZ gives solution of heat equation. Any strictly positive solution of heat equation gives solution of KPZ equation.

Thanks