An introduction to the theory of Regularity Structures

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The KPZ equation

Formally, the KPZ equation is written

$$\partial_t u = \partial_x^2 u + (\partial_x u)^2 + \xi, \qquad t \geq 0,\ x \in \mathbb{S}^1 := \mathbb{R}/2\pi\mathbb{Z}.$$

ξ is a random external force. It is not a function but a Schwartz distribution (like the derivative of a continuous non-differentiable function).

This equation was introduced in 1986 by physicists as a model for the fluctuations of a randomly growing interface and later recognised as an universal object, believed to be the limit of many discrete models in statistical mechanics.

u is expected to be similar to a Brownian motion in x, to $\partial_x u$ is not a function but a Schwartz distribution and its square is ill-defined.

Regularization of the noise and stability

Let us consider a smooth ξ_ε that converges to ξ in some sense as $\varepsilon \to 0$.

$$\partial_t u_\varepsilon = \partial_x^2 u_\varepsilon + (\partial_x u_\varepsilon)^2 + \xi_\varepsilon.$$

We are interested in the following *stability* problem: find topologies on ξ_ε and u_ε such that

1. (Probabilistic step)
   ξ_ε converges to the space-time white noise as $\varepsilon \to 0$
2. (Analytic step)
   $\xi \mapsto u$ is continuous.

Analytic step: Products of random distributions

The analytic step involves a treatment of products of random Schwartz distributions, like the term $(\partial_x u)^2$.

The theory says that under certain conditions on the SPDE (a *subcriticality* assumption in Hairer's terminology, physicists would say that the theory is *super-renormalisable*)

$$u_\varepsilon = \mathscr{F}(g_1(\xi_\varepsilon), \dots, g_k(\xi_\varepsilon))$$

where $\mathscr{F}$ is a continuous functional and $\{g_1, \dots, g_k\}$ are explicit polynomial functionals of the smooth random function ξ_ε.

For instance, if we add the term $(\partial_x u)^3$ to the equation, it is not subcritical anymore and the theory fails to apply.

The probabilistic step: Renormalisation and convergence

In the probabilistic step we need to prove that $\{g_j(\xi_\varepsilon)\}$ converge in probability.

However the $\{g_j(\cdot)\}$ are *not* continuous with respect to the very weak topology we try to impose on ξ. Convergence needs to be proved on each term separately.

In fact, some of the $\{g_j(\xi_\varepsilon)\}$ fail to converge. This is a structural problem. So what?

Some of the $g_j(\xi_\varepsilon)$ need to be modified (*renormalised*): we want to find $\{\hat{g}_j^\varepsilon(\xi_\varepsilon)\}$ which converges still retaining some link with $\{g_j(\xi_\varepsilon)\}$.

If we can do this, $\hat{u}_\varepsilon = (\hat{g}_1^\varepsilon(\xi_\varepsilon), \ldots, \hat{g}_K^\varepsilon(\xi_\varepsilon))$ converges to a *renormalised solution*.

Renormalisation of KPZ

The probabilistic step contains an additional fundamental element: the *renormalisation* procedure. For instance, in the case of the KPZ equation, the solution u_ε of

$$\partial_t u_\varepsilon = \partial_x^2 u_\varepsilon + (\partial_x u_\varepsilon)^2 + \xi_\varepsilon$$

does not converge as $\varepsilon \to 0$. It is necessary to define

$$\partial_t \hat{u}_\varepsilon = \partial_x^2 \hat{u}_\varepsilon + (\partial_x \hat{u}_\varepsilon)^2 - C_\varepsilon + \xi_\varepsilon$$

where $C_\varepsilon \to +\infty$ in order to have a convergent function $\hat{u}_\varepsilon$ as $\varepsilon \to 0$.

The term $(\partial_x \hat{u}_\varepsilon)^2 - C_\varepsilon$ converges to the *renormalised square* of $\partial_x u$.

References

- *A theory of regularity structures*, M. Hairer
  http://arxiv.org/abs/1303.5113
- *Introduction to Regularity Structures*, M. Hairer
  http://arxiv.org/abs/1401.3014
- *Singular stochastic PDEs*, M. Hairer
  http://arxiv.org/abs/1403.6353

Variables, multi-indices, monomials, derivatives

We use variables for $d \geq 1$

$$x, y, z \in \mathbb{R}^d, \qquad k, i \in \mathbb{N}^d.$$

For a multi-index $k \in \mathbb{N}^d$ we define

$$|k| := k_1 + \cdots + k_d.$$

For $k \in \mathbb{N}^d$ we define monomials

$$x^k = x_1^{k_1} \cdots x_d^{k_d}$$

with *homogeneity* $|k| := k_1 + \cdots + k_d$.

For $k \in \mathbb{N}^d$ and $\varphi \in C^\infty(\mathbb{R}^d)$ we set

$$\partial^k \varphi(x) := \frac{\partial^{k_1}}{\partial x_1^{k_1}} \cdots \frac{\partial^{k_d}}{\partial x_d^{k_d}} \varphi(x).$$

With these notations, in what follows one can (almost) forget that d is not necessarily equal to 1.

Abstract Monomials

We introduce symbols $X_1, \ldots, X_d$ and

$$X^0 := \mathbf{1}, \qquad X^k := X_1^{k_1} \cdots X_d^{k_d}, \qquad k \in \mathbb{N}^d$$

and the evaluation operators

$$\Pi_x X^k(y) = (y-x)^k.$$

Note that

$$|\Pi_x X^k(y)| \leq C\|y-x\|^{|k|}, \qquad x, y \in \mathbb{R}^d.$$

Hölder functions and abstract Taylor series

By definition, $f \in C^\gamma$, $\gamma \in \mathbb{R}_+ \setminus \mathbb{N}$, iff

$$f(y) = \sum_{|i| \le \gamma} \frac{\partial^i f(x)}{i!} (y-x)^i + r(x,y), \qquad |r(x,y)| \le C\|y-x\|^\gamma.$$

We associate to f the abstract Taylor series

$$F(x) = \sum_{|i| \le \gamma} \frac{\partial^i f(x)}{i!} X^i.$$

This is equivalent to $x \mapsto (\partial^i f(x), \, |i| \le \gamma)$.

Then :

$$\Pi_x F(x)(y) = \sum_{|i| \le \gamma} \frac{\partial^i f(x)}{i!} (y-x)^i$$

$$f(x) = \Pi_x F(x)(x), \qquad (reconstruction)$$

trivial in this case.

Γ operators

The function $f(y) := (y-z)^k$ has the abstract Taylor series

$$F(x) = \sum_{|i|\leq k} \binom{k}{i} (x-z)^{k-i} X^i = (X+x-z)^k$$

and in particular $F(z) = X^k$. We define

$$\Gamma_{xz} X^k = (X+x-z)^k = \sum_{|i|\leq k} \binom{k}{i} (x-z)^{k-i} X^i.$$

Γ_{xz} is a rule to transform a Taylor sum around z in one around x.

This definition satisfies the simple properties $\Gamma_{xy}\Gamma_{yz} = \Gamma_{xz}$, $\Gamma_{xx} = Id$,

$$\Pi_z = \Pi_x \Gamma_{xz}, \quad |\Gamma_{xz} X^k - X^k| < k, \quad \|\Gamma_{xz} X^k - X^k\|_i \leq C\|x-z\|^{k-i}.$$

Hölder functions

To $f \in C^\gamma$, $\gamma \in \mathbb{R}_+ \setminus \mathbb{N}$, we associate

$$F(x) = \sum_{|i| \le \gamma} \frac{\partial^i f(x)}{i!} X^i.$$

Then

$$F(x) - \Gamma_{xz} F(z) = \sum_{|i| \le \gamma} \frac{X^i}{i!} \left(\partial^i f(x) - \sum_{|j| \le \gamma - |i|} \frac{\partial^{i+j} f(z)}{j!} (x-z)^j \right)$$

and in particular $f \in C^\gamma$ iff $\partial^i f \in C^{\gamma - |i|}$ for all $|i| \le \gamma$, i.e.

$$\| F(x) - \Gamma_{xz} F(z) \|_i \le C \| x - z \|^{\gamma - |i|}.$$

Hölder functions

To $f \in C^\gamma$, $\gamma \in \mathbb{R}_+ \setminus \mathbb{N}$, we associate

$$F(x) = \sum_{|i| \leq \gamma} \frac{\partial^i f(x)}{i!} X^i.$$

Then

$$F(x) - \Gamma_{xz} F(z) = \sum_{|i| \leq \gamma} \frac{X^i}{i!} \left(\partial^i f(x) - \sum_{|j| \leq \gamma - |i|} \frac{\partial^{i+j} f(z)}{j!} (x-z)^j \right)$$

and in particular $f \in C^\gamma$ iff $\partial^i f \in C^{\gamma - |i|}$ for all $|i| \leq \gamma$, i.e.

$$\|F(x) - \Gamma_{xz} F(z)\|_i \leq C \|x - z\|^{\gamma - |i|}.$$

We want to add to this classical framework generalised monomials: these will be *random (Schwartz) distributions*.

Schwartz distributions

The *Schwartz space* or *space of rapidly decreasing functions* on $\mathbb{R}^d$ is

$$\mathcal{S}(\mathbb{R}^d) := \{\varphi \in C^\infty(\mathbb{R}^d) : p_{\alpha,\beta}(\varphi) < +\infty, \ \forall \alpha, \beta \in \mathbb{N}^d\}$$

$$p_{\alpha,\beta}(\varphi) := \sup_{x \in \mathbb{R}^d} |x^\alpha \partial^\beta \varphi(x)|, \qquad \alpha, \beta \in \mathbb{N}^d.$$

A *tempered* (or *Schwartz*) *distribution* is a linear functional $T : \mathcal{S}(\mathbb{R}^d) \mapsto \mathbb{R}$ such that there exist a constant C and $\alpha_1, \beta_1, \ldots, \alpha_k, \beta_k \in \mathbb{N}^d$ s.t.

$$|T(\varphi)| \le C \sum_{i=1}^{k} p_{\alpha_i,\beta_i}(\varphi), \qquad \forall \, \varphi \in \mathcal{S}(\mathbb{R}^d).$$

We write $T \in \mathcal{S}'(\mathbb{R}^d)$. Examples:

- If $f \in L^p(\mathbb{R}^d)$, $p \ge 1$, then $T(\varphi) := \int \varphi f \, \mathrm{d}x$
- For any finite measure μ on $\mathbb{R}^d$, $T(\varphi) := \int \varphi \, \mathrm{d}\mu$
- For any finite measure μ on $\mathbb{R}^d$ and $\alpha \in \mathbb{N}^d$, $T(\varphi) := \int \partial^\alpha \varphi \, \mathrm{d}\mu$
- Linear combinations of the above examples

Generalised monomials

We consider a class of symbols $\mathcal{T} \supseteq \{X^k, k \geq 0\}$ representing generalised monomials. We associate to each $\tau \in \mathcal{T}$ a *degree* or *homogeneity* :

$$\forall \tau \in \mathcal{T}, \qquad |\tau| \in \mathbb{R},$$

with the assumption that $|X^k| = |k| = k_1 + \cdots + k_d$.

We want to endow $\mathcal{T}$ with operators (Π_x, Γ_{xz}) such that:

1. Π_x associates to each symbol $\tau \in \mathcal{T}$ a (random) distribution on $\mathbb{R}^d$ whose local regularity around each point x is no worse than the homogeneity $|\tau|$; informally:

$$|\Pi_x \tau(y)| \leq C \|y - x\|^{|\tau|}, \qquad x, y \in \mathbb{R}^d$$

2. Γ_{xz} is a rule to transform an abstract Taylor sum around z in one around x.

General models

A model of $\mathcal{T}$ is given by a couple (Π_x, Γ_{xz}) such that

- for all x, $\Pi_x : \mathcal{T} \mapsto \mathcal{S}'(\mathbb{R}^d)$ and for all $\varphi \in C_c^\infty(\mathbb{R}^d)$

  $$|\Pi_x \tau(\varphi_{x,\delta})| \leq C\delta^{|\tau|},$$

  where $\varphi_{x,\delta}(y) := \delta^{-d}\varphi\left(\frac{y-x}{\delta}\right)$, $\delta > 0$.
- for all x, y, z, $\Gamma_{xz} : \langle\mathcal{T}\rangle \mapsto \langle\mathcal{T}\rangle$ is such that $\Gamma_{xx} = Id$, $\Gamma_{xy}\Gamma_{yz} = \Gamma_{xz}$ and

  $$|\Gamma_{xz}\tau - \tau| < |\tau|, \qquad \|\Gamma_{xz}\tau - \tau\|_\alpha \leq C\|z - x\|^{|\tau|-\alpha}, \ \alpha < |\tau|.$$

- for all x, z: $\Pi_z = \Pi_x \Gamma_{xz}$.

Very important: (Π_x, Γ_{xz}) is in general random (measurable with respect to ξ or ξ_ε), $\mathcal{T}$ is fixed and deterministic.

A 1-dimensional example

Let us add a symbol $\mathbb{W}$ to $\mathcal{T}$. Let $(B_t)_{t\in\mathbb{R}}$ be a two-sided Brownian motion. Then a possible choice for Π_x and Γ_{xz} is

$$\Pi_x \mathbb{W}(y) = B_y - B_x, \qquad \Gamma_{xz}\mathbb{W} = \mathbb{W} + (B_x - B_z)\mathbf{1}.$$

$$|\mathbb{W}| = \frac{1}{2} - \varepsilon =: \alpha, \qquad |\Pi_x \mathbb{W}(y)| = |B_y - B_x| \le C|y-x|^{\alpha}$$

$$\begin{aligned} \Pi_x \Gamma_{xz} \mathbb{W}(y) &= \Pi_x(\mathbb{W} + (B_x - B_z)\mathbf{1})(y) = B_y - B_x + B_x - B_z \\ &= B_y - B_z = \Pi_z \mathbb{W}(y) \end{aligned}$$

$$\|\Gamma_{xz}\mathbb{W} - \mathbb{W}\|_0 = |B_x - B_z| \le C|x-z|^{\alpha - 0}$$

A 1-dimensional example

Let us add yet another symbol Ξ to $\mathcal{T}$. A possible choice for Π_x and Γ_{xz} is for $\varphi \in C_c^\infty(\mathbb{R})$

$$\Pi_x \Xi(\varphi) = \int_{\mathbb{R}} \varphi(y)\, \mathrm{d}B_y = -\int_{\mathbb{R}} \varphi'(y)\,(B_y - B_x)\,\mathrm{d}y, \qquad \Gamma_{xz}\Xi = \Xi.$$

The homogeneity of Ξ is $|\Xi| = \alpha - 1 = -\frac{1}{2} - \varepsilon$. It is easy to show that the required assumptions are satisfied:

$$|\Pi_x \Xi(\varphi_{x,\delta})| = \left| \int_{\mathbb{R}} (B_y - B_x)\, \delta^{-2} \varphi' \left(\frac{y-x}{\delta} \right) \mathrm{d}y \right| \le C \delta^{-\frac{1}{2}-\varepsilon}$$

Hölder functions

For $\gamma > 0$ we say that $F \in \mathcal{D}^\gamma$ if F takes values in the linear span of the symbols with homogeneity $< \gamma$ and for all $\alpha < \gamma$

$$\|F(x) - \Gamma_{xz} F(z)\|_\alpha \leq C\|x - z\|^{\gamma-\alpha}$$

where $\|\cdot\|_\alpha$ is the norm of the projection onto the span of the symbols with homogeneity equal to α. (Note that α is not restricted to $\mathbb{N}$ anymore!).

The inspiration comes from Gubinelli's theory of *controlled rough paths*.

A 1-dimensional example

Let $\sigma : \mathbb{R} \mapsto \mathbb{R}$ be smooth. Then

$$|\sigma(B_x) - \sigma(B_z)| \leq C|B_x - B_z| \leq C|x - z|^\alpha.$$

This means $F(x) = \sigma(B_x)\mathbf{1}$ defines a function $F \in \mathcal{D}^\alpha$, since

$$F(x) - \Gamma_{xz}F(z)$$

$$\|F(x) - \Gamma_{xz}F(z)\|_0 \leq C|x - z|^{\alpha - 0}$$

Can we go beyond α?

A 1-dimensional example

By the Ito formula

$$\sigma(B_x) = \sigma(B_z) + \int_z^x \sigma'(B_u)\,\mathrm{d}B_u + \frac{1}{2}\int_z^x \sigma''(B_u)\,\mathrm{d}u.$$

By a (nontrivial) result similar to the Kolmogorov criterion and due to M. Gubinelli

$$\left|\int_z^x (\sigma'(B_u) - \sigma'(B_z))\,\mathrm{d}B_u\right| \le C|x-z|^{2\alpha}.$$

Then

$$\left|\sigma(B_x) - \sigma(B_z) - \sigma'(B_z)(B_x - B_z)\right| \le C|x-z|^{2\alpha}$$

A 1-dimensional example

Then let us set

$$G(x) = \sigma(B_x)\mathbf{1} + \sigma'(B_x)\mathbb{W}.$$

Then

$$G(x) - \Gamma_{xz}G(z) =$$
$$= \left[\sigma(B_x) - \sigma(B_z) - \sigma'(B_z)(B_x - B_z)\right]\mathbf{1} + (\sigma'(B_x) - \sigma'(B_z))\mathbb{W}$$

and $G \in \mathcal{D}^{2\alpha}$:

$$\|G(x) - \Gamma_{xz}G(z)\|_\beta \le C|x - z|^{2\alpha-\beta}, \qquad \beta \in \{0, \alpha\}.$$

Note that:

1. F is a truncation of G
2. If $\xi \in C^\beta$ with $\beta \ge 2\alpha$, then $H(x) = (\sigma(B_x) + \xi_x)\mathbf{1} + \sigma'(B_x)\mathbb{W}$ defines a function $H \in \mathcal{D}^{2\alpha}$

Hölder functions

In the general case, for $\gamma > 0$ we say that $F \in \mathcal{D}^\gamma$ if F takes values in the linear span of the symbols with homogeneity $< \gamma$ and for all $\alpha < \gamma$

$$\|F(x) - \Gamma_{xz} F(z)\|_\alpha \leq C\|x - z\|^{\gamma-\alpha}$$

where $\|\cdot\|_\alpha$ is the norm of the projection onto the span of the symbols with homogeneity equal to α. (Note that α is not restricted to $\mathbb{N}$ anymore!).

Given a Taylor expansion $F(x)$ around each point x, can we find a function/distribution f which has this expansion up to a remainder?

It $f \in C^\gamma$, $\gamma \in \mathbb{R}_+ \setminus \mathbb{N}$, and $F(x) = \sum_{|i|\leq\gamma} \frac{\partial^i f(x)}{i!} X^i$, then $F \in \mathcal{D}^\gamma$ and $f(x) = \Pi_x F(x)(x)$. But in general $\Pi_x F(x)(\cdot)$ is a distribution and we can not compute it at x.

The reconstruction theorem

Reconstruction Theorem. *For all* $\gamma > 0$ *and* $F \in \mathcal{D}^\gamma$ *there exists a unique distribution* $\mathcal{R}F$ *on* $\mathbb{R}^d$ *such that*

$$|\mathcal{R}F(y) - \Pi_x F(x)(y)| \leq C\|x - y\|^\gamma$$

or, more precisely, such that

$$|\mathcal{R}F(\varphi_{x,\delta}) - \Pi_x F(x)(\varphi_{x,\delta})| \leq C\delta^\gamma.$$

Again for all $\varphi \in C_c^\infty(\mathbb{R}^d)$, $\varphi_{x,\delta}(y) := \delta^{-d}\varphi\left(\frac{y-x}{\delta}\right)$, $\delta > 0$.

$\Pi_x F(x)(\cdot)$ being a distribution, we can not set $\mathcal{R}F(x) = \Pi_x F(x)(x)$.

The proof is based on Wavelets Analysis, like for Paraproducts in Massimiliano's course.

How to define $a_x \, \mathrm{d}B_x$

Let us consider a continuos process a_x, $x \in \mathbb{R}$, and a deterministic $\varphi \in C_c^\infty(\mathbb{R})$. Can we define $\int_{\mathbb{R}} \varphi(x) \, a_x \, \mathrm{d}B_x$ pathwise?

Idea: use the reconstruction theorem!

If we define	$F(x) := a_x \Xi$
then	$\Pi_x F(x)(y) = a_x \, \mathrm{d}B_y$
and formally	$\mathcal{R}F(x) = \Pi_x F(x)(x) = a_x \, \mathrm{d}B_x.$

But we need $F \in \mathcal{D}^\gamma$ with $\gamma > 0$.

How to define $a_x \, \mathrm{d}B_x$

$F(x) := a_x \Xi$, $F \in \mathcal{D}^\gamma$ with $\gamma > 0$ depends on

$$F(x) - \Gamma_{xz} F(z) = a_x \Xi - \Gamma_{xz} a_z \Xi = (a_x - a_z)\Xi$$

satisfying $\|F(x) - \Gamma_{xz} F(z)\|_i \le C|x-z|^{\gamma - i}$ for all $i \le \gamma$ i.e.

$$|a_x - a_z| \le C|x-z|^{\gamma+1-\alpha} \qquad (|\Xi| = \alpha - 1 = -1/2 - \varepsilon).$$

Then $F \in \mathcal{D}^\gamma$ with $\gamma > 0$ if $a \in C^{\gamma+1-\alpha}$ with $\gamma + 1 - \alpha > 1/2 + \varepsilon$. This is in fact a Theorem due to Young (1936).

However this is unsatisfactory since it rules out SDEs like $X_t = X_0 + \int_0^t \sigma(X_s) \, \mathrm{d}B_s$.

The symbol for $B_y \, \mathrm{d}B_y$

Let us add another symbol $\Xi\mathbb{W}$ to $\mathcal{T}$.

$$\Pi_x \Xi\mathbb{W}(\varphi) = \int_{\mathbb{R}} \varphi(y)(B_y - B_x) \, \mathrm{d}B_y \qquad \text{(Ito integral)}$$

$$= -\int_{\mathbb{R}} \frac{1}{2} \left((B_y - B_x)^2 - (y - x) \right) \varphi'(y) \, \mathrm{d}y$$

$$\Gamma_{xz} \Xi\mathbb{W} = \Xi(\mathbb{W} + (B_x - B_z)\mathbf{1}) = \Xi\mathbb{W} + (B_x - B_z)\Xi.$$

$$\Pi_x \Gamma_{xz} \Xi\mathbb{W} = \int_{\mathbb{R}} \varphi(y)(B_y - B_x) \, \mathrm{d}B_y + \int_{\mathbb{R}} \varphi(y)(B_x - B_z) \, \mathrm{d}B_y$$

$$|\Xi\mathbb{W}| := |\Xi| + |\mathbb{W}| = 2\alpha - 1 = -2\varepsilon.$$

$$|\Pi_x \Xi\mathbb{W}(\varphi_{x,\delta})| = \left| \int_{\mathbb{R}} \frac{1}{2} \left((B_y - B_x)^2 - (y - x) \right) \delta^{-2} \varphi' \left(\frac{y - x}{\delta} \right) dy \right|$$

$$\leq C \delta^{-2\varepsilon}$$

How to define $a_x \, dB_x$

Let us now try $F(x) := a_x \Xi + b_x \Xi \mathbb{W}$,

$$F(x) - \Gamma_{xz} F(z) = (a_x - a_z - b_z(B_x - B_z))\Xi + (b_x - b_z)\Xi\mathbb{W}$$

and we must have $\|F(x) - \Gamma_{xz} F(z)\|_i \leq C|x - z|^{\gamma - i}$ i.e.

$$|a_x - a_z - b_z(B_x - B_z)| \leq C|x - z|^{\gamma+1-\alpha} \qquad (|\Xi| = \alpha - 1)$$

$$|b_x - b_z| \leq C|x - z|^{\gamma+1-2\alpha} \qquad (|\Xi\mathbb{W}| = 2\alpha - 1)$$

How to define $a_x \, \mathrm{d}B_x$

If we set $A(x) := a_x \mathbf{1} + b_x \mathbb{W}$, then

$$A(x) - \Gamma_{xz} A(z) = (a_x - a_z - b_z(B_x - B_z))\mathbf{1} + (b_x - b_z)\mathbb{W}$$

we know from the previous slide that

$$|a_x - a_z - b_z(B_x - B_z)| \le C|x - z|^{\gamma+1-\alpha} \qquad (|\mathbf{1}| = 0)$$

$$|b_x - b_z| \le C|x - z|^{\gamma+1-2\alpha} \qquad (|\mathbb{W}| = \alpha)$$

and this means $A \in \mathcal{D}^{\gamma+1-\alpha}$. Then

$$A \in \mathcal{D}^{\gamma+1-\alpha} \iff F = A\Xi \in \mathcal{D}^{\gamma}.$$

Note that $|\Xi| = \alpha - 1$.

How to define $a_x \, \mathrm{d}B_x$

Let us fix $\alpha \in]1/3, 1/2[$ and set $\gamma = 3\alpha - 1$, so that $\gamma + 1 - \alpha = 2\alpha$.

We assume that $A(x) := a_x \mathbf{1} + b_x \mathbb{W}$ belongs to $\mathcal{D}^{2\alpha}$.

Moreover $x \mapsto H(x) := \Xi$ belongs to $\mathcal{D}^{\infty}$ since $H(x) - \Gamma_{xz} H(z) \equiv 0$.

Note that $\mathcal{R}A = a$, $\mathcal{R}H = \mathrm{d}B$. We have shown that

$$A * H(x) = F(x) = a_x \Xi + b_x \Xi \mathbb{W}$$

belongs to $\mathcal{D}^{3\alpha-1}$ with $3\alpha - 1 > 0$. The reconstruction theorem yields $\mathcal{R}F =: a_y \, \mathrm{d}B_y$ such that for all x

$$\left| \int_{\mathbb{R}} \varphi_{x,\delta} \, a \, \mathrm{d}B - a_x \int_{\mathbb{R}} \varphi_{x,\delta} \, \mathrm{d}B - b_x \int_{\mathbb{R}} \varphi_{x,\delta} \, (B - B_x) \, \mathrm{d}B \right| \leq C \delta^{3\alpha-1}.$$

This is a result due to M. Gubinelli (2004).

An exercise

Let $\gamma > \beta$. For all $U \in \mathcal{D}^\gamma$ with

$$U(x) = \sum_{|\tau|<\gamma} u_\tau(x)\tau$$

set

$$Q_\beta^- U(x) = \sum_{|\tau|<\beta} u_\tau(x)\tau$$

and show that $Q_\beta^- U \in \mathcal{D}^\beta$.

If moreover $\gamma > \beta > 0$ then show that $\mathcal{R}U = \mathcal{R}Q_\beta^- U$.

Another exercise

Suppose that $0 < \alpha < \gamma$, $U \in \mathcal{D}^\gamma$ and U takes values in the linear span of $\{X^k, k\} \cup V$ with

$$|\tau| \geq \alpha, \qquad \forall \tau \in V.$$

Let $V(x)$ be the projection of $U(x)$ on the linear span of $\{X^k, k\}$.

Then show that

1. $V \in \mathcal{D}^\alpha$,
2. the function $x \mapsto u(x) := \langle U(x), \mathbf{1} \rangle$ is in C^α
3. $\mathcal{R}U = \mathcal{R}V = u$.

Some notations: the heat kernel

For $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ we define the heat kernel $G : \mathbb{R}^d \mapsto \mathbb{R}$

$$G(x) = \mathbb{1}_{(x_d>0)} \frac{1}{(2\pi x_d)^{\frac{d-1}{2}}} \exp\left(-\frac{x_1^2 + \cdots + x_{d-1}^2}{2x_d}\right).$$

Given $k = (k_1, \ldots, k_d) \in \mathbb{N}^d$ we define

$$G^{(k)}(x) = \frac{\partial^{k_1}}{\partial x_1^{k_1}} \cdots \frac{\partial^{k_d}}{\partial x_d^{k_d}} G(x).$$

Parabolic scaling

The heat kernel has a very important scaling property:

$$G(\delta x_1, \dots, \delta x_{d-1}, \delta^2 x_d) = \frac{1}{\delta} G(x_1, \dots, x_{d-1}, x_d), \qquad \delta > 0$$

which is associated with the scaling $\mathfrak{s} := (1, \dots, 1, 2)$.

This motivates the following definitions:

$$\|x - y\|_{\mathfrak{s}} := |x_1 - y_1| + \cdots + |x_{d-1} - y_{d-1}| + |x_d - y_d|^{1/2},$$

$$|k|_{\mathfrak{s}} := k_1 + \cdots + k_{d-1} + 2k_d, \qquad k \in \mathbb{N}^d.$$

Regularity Structures

A *regularity structure* $\mathscr{T} = (A, T, G)$ consists of the following elements:

- An index set $A \subset \mathbb{R}$ such that $0 \in A$, A is bounded below and A has no finite accumulation point
- A graded vector space $T = \bigoplus_{\alpha \in A} T_\alpha$, with each T_α a Banach space. Furthermore $T_0 \approx \mathbb{R}\mathbf{1}$.
- A *structure group* G of linear operators acting on T such that $\forall\, \Gamma \in G$, $\alpha \in A$, $a \in T_\alpha$

$$\Gamma a - a \in \bigoplus_{\beta < \alpha} T_\beta .$$

Furthermore $\Gamma \mathbf{1} = \mathbf{1}$ for all $\Gamma \in G$.

Models

A model of $\mathscr{T} = (A, T, G)$ with scaling $\mathfrak{s}$ is given by a couple (Π_x, Γ_{xz}) such that

- for all x, $\Pi_x : T \mapsto \mathcal{S}'(\mathbb{R}^d)$ and for all $\varphi \in C_c^\infty(\mathbb{R}^d)$

$$|\Pi_x \tau(\varphi_{x,\delta})| \leq C\delta^\ell, \qquad \forall \tau \in T_\ell$$

  where $\varphi_{x,\delta}(y) := \delta^{-d-1}\varphi\left(\frac{y_1-x_1}{\delta}, \ldots, \frac{y_{d-1}-x_{d-1}}{\delta}, \frac{y_d-x_d}{\delta^2}\right)$, $\delta > 0$.
- a map $\Gamma : \mathbb{R}^d \times \mathbb{R}^d \mapsto G$ such that for all x, y, z, $\Gamma_{xx} = Id$, $\Gamma_{xy}\Gamma_{yz} = \Gamma_{xz}$ and

$$\|\Gamma_{xz}\tau - \tau\|_\alpha \leq C\|z - x\|_{\mathfrak{s}}^{\ell-\alpha}, \qquad \forall \tau \in T_\ell.$$

- for all x, z: $\Pi_z = \Pi_x \Gamma_{xz}$.

The reconstruction theorem

For $\gamma > 0$ we say that $F \in \mathcal{D}^\gamma$ if $F : \mathbb{R}^d \mapsto \bigoplus_{\alpha<\gamma} T_\alpha$ and for all $\alpha < \gamma$

$$\|F(x) - \Gamma_{xz} F(z)\|_{T_\alpha} \le C\|x - z\|_{\mathfrak{s}}^{\gamma-\alpha}$$

where $\|\cdot\|_{T_\alpha}$ is the norm of the projection onto T_α.

Reconstruction Theorem. *For all $\gamma > 0$ and $F \in \mathcal{D}^\gamma$ there exists a unique $\mathcal{R}F \in \mathcal{S}'(\mathbb{R}^d)$ such that*

$$|\mathcal{R}F(y) - \Pi_x F(x)(y)| \le C\|x - y\|_{\mathfrak{s}}^{\gamma}$$

or, more precisely, such that

$$|\mathcal{R}F(\varphi_{x,\delta}) - \Pi_x F(x)(\varphi_{x,\delta})| \le C\delta^\gamma.$$

The mild formulation

Now, how can we use this formalism to solve SPDEs?

Let us go back to the regularized (unrenormalised) KPZ

$$\partial_t u_\varepsilon = \partial_x^2 u_\varepsilon + (\partial_x u_\varepsilon)^2 + f(u_\varepsilon) + \xi_\varepsilon.$$

with an additional non-linear term, for $f : \mathbb{R} \mapsto \mathbb{R}$ smooth.

Here everything is classical, and the solution is given by the fixed point of the *mild formulation*

$$u_\varepsilon(t,x) = \int_{[0,t]\times\mathbb{S}^1} G_{t-s}(x-y)\left((\partial_x u_\varepsilon)^2 + f(u_\varepsilon) + \xi_\varepsilon\right)(s,y)\,\mathrm{d}s\,\mathrm{d}y$$

(we assume for simplicity that $u_\varepsilon(0,\cdot) \equiv 0$).

Lift ot the equation

Now we want to consider the equation

$$u_\varepsilon = G * \left((\partial_x u_\varepsilon)^2 + f(u_\varepsilon) + \xi_\varepsilon \right),$$

build a regularity structure $\mathscr{T} = (A, T, G)$ with a model $(\Pi^\varepsilon, \Gamma^\varepsilon)$, and find a $U_\varepsilon \in \mathcal{D}^\gamma = \mathcal{D}^\gamma(\Pi^\varepsilon, \Gamma^\varepsilon)$ such that

$$\mathcal{R} U_\varepsilon = u_\varepsilon,$$

$$U_\varepsilon = \mathcal{K}(\mathscr{D} U_\varepsilon \cdot \mathscr{D} U_\varepsilon + \mathcal{F}(U_\varepsilon) + \Xi)$$

- $\mathcal{K} : \mathcal{D}^\gamma \mapsto \mathcal{D}^{\gamma+2}$, $\quad \mathcal{R}(\mathcal{K} V) = G * \mathcal{R} V$
- $\mathscr{D} : \mathcal{D}^\gamma \mapsto \mathcal{D}^{\gamma-1}$, $\quad \mathcal{R}(\mathscr{D} V) = \partial_x \mathcal{R} V$
- $\mathcal{F} : \mathcal{D}^\gamma \mapsto \mathcal{D}^\gamma$, $\quad \mathcal{R}(\mathcal{F} V) = f(\mathcal{R} V)$
- and finally there is the $\cdot$

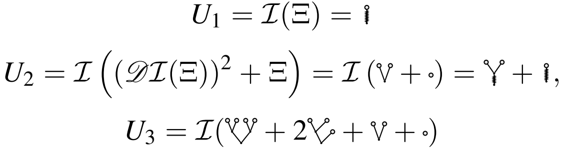
How to build the regularity structure (A, T, G)

We shall need $\Xi \in T$. We also use the notation $\Xi = \circ$

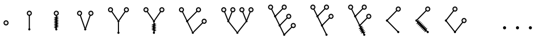
Let us write $U_0 = 0$ and (forgetting the non-linear term $f(u)$)

$$U_{n+1} = \mathcal{I}(\mathscr{D}U_n \cdot \mathscr{D}U_n + \Xi).$$

Then



In this way we generate iteratively a list of symbols (trees):



Abstract Monomials

We define the following family $\mathcal{F}$ of symbols:

- $1, X_1, \dots, X_d, \Xi \in \mathcal{F}$
- if $\tau_1, \dots, \tau_n \in \mathcal{F}$ then $\tau_1 \cdots \tau_n \in \mathcal{F}$ (commutative and associative product)
- if $\tau \in \mathcal{F}$ then $\mathcal{I}(\tau) \in \mathcal{F}$ and $\mathcal{I}_k(\tau) \in \mathcal{F}$ (formal convolution with the heat kernel differentiated $k \in \mathbb{N}^d$ times), with $\mathcal{I}(X^k) = 0$

Examples: $\mathcal{I}(\Xi), X^n \Xi \mathcal{I}_k(\Xi), \mathcal{I}((\mathcal{I}_1(\Xi))^2), \Xi^n$

To a symbol τ we associate a real number $|\tau|_\mathfrak{s}$ called its homogeneity:

$$|\Xi|_\mathfrak{s} < -\frac{d+1}{2}, \quad |X_1|_\mathfrak{s} = \cdots = |X_{d-1}|_\mathfrak{s} = 1, \quad |X_d|_\mathfrak{s} = 2, \quad |1| = 0$$

$$|\tau_1 \cdots \tau_n|_\mathfrak{s} = |\tau_1|_\mathfrak{s} + \cdots + |\tau_n|_\mathfrak{s}, \quad |\mathcal{I}_k(\tau)|_\mathfrak{s} = |\tau|_\mathfrak{s} + 2 - |k|_\mathfrak{s}.$$

However $\mathcal{F}$ is far too large. We do not need Ξ^n for $n \geq 2$.

Sub-criticality assumption

We consider a SPDE $\partial_t u = \Delta u + F(u, \nabla u, \xi)$ on $\mathbb{R}^d$. In the formal expression for F:

- we replace ξ by Ξ, with $|\Xi| = \alpha < 0$
- if $\alpha + 1 < 0$ then we associate to $\partial_i u$ a symbol P_i, $|P_i| = \alpha + 1$, $i = 1, \ldots, d-1$
- if $\alpha + 2 < 0$ then we associate to u a symbol U, $|U| = \alpha + 2$.

We assume that

1. The resulting expression is polynomial in the symbols.
2. Terms containing Ξ do not contain other symbols, and all other monomials have homogeneity $> \alpha$.

Then we say that the SPDE is *sub-critical*.

Examples of sub-critical equations

KPZ in 1 space-dim. ($d = 2$). Here $\alpha = |\Xi| = -\frac{3}{2} - \delta$.

$$\partial_t u = \Delta u + F(u, \nabla u, \xi), \qquad F(u, p, \xi) = p^2 + \xi.$$

Then the formal replacement gives $F(U, P, \Xi) = P^2 + \Xi$, $|P| = \alpha + 1$, $|P^2| = 2\alpha + 2 > \alpha$ since $\alpha > -2$.

If we modify the equation

$$\partial_t u = \Delta u + F(u, \nabla u, \xi), \qquad F(u, p, \xi) = p^3 + \xi.$$

Then the formal replacement gives $F(U, P, \Xi) = P^3 + \Xi$, $|P| = \alpha + 1$, $|P^3| = 3\alpha + 3 < \alpha$ since $\alpha < -\frac{3}{2}$.

Then this class of equations is sub-critical for $F(u, p, \xi) = p^2 + \xi$ and critical or super-critical for $F(u, p, \xi) = p^k + \xi$, $k \geq 3$.

Examples of sub-critical equations

Parabolic Anderson model. Here the space-dim. is 2 ($d = 3$) and ξ is only a function of space. Therefore $\alpha = -1 - \delta$. The equation is

$$\partial_t u = \Delta u + F(u, \xi), \qquad F(u\xi) = u\xi.$$

Then the formal replacement gives $F(u, \Xi) = u\Xi$, $\alpha + 2 > 0$.

Another important example is the Φ^4_3 model: space-dim. is 3 ($d = 4$), $\alpha = -\frac{5}{2} - \delta$,

$$\partial_t u = \Delta u + F(u, \xi), \qquad F(u, \xi) = u^3 + \xi.$$

Then the formal replacement gives $F(U, \Xi) = U^3 + \Xi$, $|U| = \alpha + 2 < 0$, $|U^3| = 3\alpha + 6 > \alpha$ since $\alpha < -\frac{3}{2}$.

Then this class of equations is sub-critical for $F(u, \xi) = u^4 + \xi$ and critical or super-critical for $F(u, \xi) = u^k + \xi$, $k \geq 5$.

Construction of the regularity structure

Now we define a set $\mathfrak{M}_F$ of monomials in the above symbols: $\Xi^m U^n P^k \in \mathfrak{M}_F$ if F contains a term $\xi^{\overline{m}} u^{\overline{n}} (Du)^{\overline{k}}$ with $m \leq \overline{m} \leq 1$, $n \leq \overline{n}$, $k \leq \overline{k}$.

For KPZ: $F(u,p,\xi) = p^2 + \xi, \quad \mathfrak{M}_F = \{\Xi, P, P^2\}$.

For PAM: $F(u,p,\xi) = u\xi, \quad \mathfrak{M}_F = \{U, U\Xi, \Xi\}$.

For Φ^4_3: $F(u,p,\xi) = u^3 + \xi, \quad \mathfrak{M}_F = \{U, U, U^3, \Xi\}$.

Construction of the regularity structure

We set $\mathcal{W}_0 = \mathcal{U}_0 = \mathcal{P}_0^i = \emptyset$ and, given $A, B \subset \mathcal{F}$, we also write

$$AB := \{\tau\bar{\tau} : \tau \in A, \bar{\tau} \in B\}.$$

Then, we define the sets $\mathcal{W}_n, \mathcal{U}_n$ and $\mathcal{P}_n^i$ for $n > 0$ recursively by

$$\mathcal{W}_n = \mathcal{W}_{n-1} \cup \bigcup_{\mathcal{Q} \in \mathfrak{M}_F} \mathcal{Q}(\mathcal{U}_{n-1}, \mathcal{P}_{n-1}, \Xi) ,$$
$$\mathcal{U}_n = \{X^k\} \cup \{\mathcal{I}(\tau) : \tau \in \mathcal{W}_n\} ,$$
$$\mathcal{P}_n^i = \{X^k\} \cup \{\mathcal{I}_i(\tau) : \tau \in \mathcal{W}_n\} ,$$

and finally

$$\mathcal{F}_F := \bigcup_{n \geq 0} (\mathcal{W}_n \cup \mathcal{U}_n).$$

Construction of the regularity structure

Lemma *The set* $\{\tau \in \mathcal{F}_F : |\tau|_\mathfrak{s} \leq \gamma\}$ *is finite for all* $\gamma \in \mathbb{R}$ *iff the SPDE is sub-critical.*

Then we set $A := \{|\tau|_\mathfrak{s} : \ \tau \in \mathcal{F}_F\}$,

$$T_\gamma := \{\lambda_1 \tau_1 + \cdots + \lambda_k \tau_k : \ |\tau_i|_\mathfrak{s} = \gamma\}.$$

By the Lemma each T_γ is finite-dimensional (if non-empty).

The group structure G will be defined below.

Another example: Φ^4_3

We have seen the family of symbols (trees) of KPZ.

For Φ^4_3 we have with the notation

$$\Xi = \circ\,, \quad \mathcal{I}(\Xi) = \text{⫯}\,, \quad \mathcal{I}(\Xi)^3 = \Psi\,, \quad \mathcal{I}(\Xi)\mathcal{I}(\mathcal{I}(\Xi)^3) = \text{tree}_4\,,$$

the list of symbols (trees)

$$T = \langle \circ\,, \vee, \text{tree}_2, \text{tree}_5, \text{⫯}, \text{tree}_4, \text{tree}_3, X_i \text{tree}_2, \mathbf{1}, \text{tree}_3, \text{tree}_3, \dots \rangle\ .$$

The Π_x operators

We fix a (continuous) path of ξ_ε and define recursively (continuous) generalized monomials $\Pi_x^\varepsilon \tau$ around the base point x

$$\Pi_x^\varepsilon X_i(y) = (y_i - x_i), \qquad \Pi_x^\varepsilon \Xi(y) = \xi_\varepsilon(y),$$

$$\Pi_x^\varepsilon(\tau_1 \cdots \tau_n)(y) = \prod_{i=1}^n \Pi_x^\varepsilon \tau_i(y),$$

$$\Pi_x^\varepsilon \mathcal{I}_k(\tau)(y) = (G^{(k)} * \Pi_x^\varepsilon \tau)(y) - \sum_{|i|_\mathfrak{s} < |\mathcal{I}_k(\tau)|_\mathfrak{s}} \frac{(y-x)^i}{i!} (G^{(i+k)} * \Pi_x^\varepsilon \tau)(x)$$

Then we can interpret analytically $|\tau|_\mathfrak{s}$

$$|\Pi_x^\varepsilon \tau(y)| \le C_\varepsilon \|y - x\|_\mathfrak{s}^{|\tau|_\mathfrak{s}}.$$

The Γ operators

We have the recursive definition of $\Gamma^\varepsilon_{xz} : T \mapsto T$

$$\Gamma^\varepsilon_{xz} X_i = X_i + (x_i - z_i), \qquad \Gamma^\varepsilon_{xz} \Xi = \Xi, \qquad \Gamma^\varepsilon_{xz} \prod_i \tau_i = \prod_i \Gamma^\varepsilon_{xz} \tau_i$$

$$\Gamma^\varepsilon_{xz} \mathcal{I}_k(\tau) = \mathcal{I}_k(\Gamma^\varepsilon_{xz}\tau) - \sum_{|j|_\mathfrak{s} < |\tau|_\mathfrak{s} + 2 - |k|} (\Pi^\varepsilon_x \mathcal{I}_{k+j}(\Gamma^\varepsilon_{xz}\tau))(z) \frac{(X + x - z)^j}{j!}$$

On can check again the compatibility condition

$$\Pi^\varepsilon_z = \Pi^\varepsilon_x \Gamma^\varepsilon_{xz}$$

and the properties $\Gamma^\varepsilon_{xx} = Id$, $\Gamma^\varepsilon_{xy}\Gamma^\varepsilon_{yz} = \Gamma^\varepsilon_{xz}$,

$$|\Gamma^\varepsilon_{xz}\tau - \tau|_\mathfrak{s} < |\tau|_\mathfrak{s}, \qquad \|\Gamma^\varepsilon_{xz}\tau - \tau\|_\ell \leq C\|x - z\|_\mathfrak{s}^{|\tau| - \ell}, \ \ell < |\tau|_\mathfrak{s}.$$

Examples

Let for instance $d = 2$. Then $\alpha = |\Xi|_{\mathfrak{s}} = -\frac{3}{2} - \delta$, $\delta > 0$.

$$\Pi_x^\varepsilon \mathcal{I}(\Xi)(y) = (G * \xi_\varepsilon)(y) - (G * \xi_\varepsilon)(x)$$

The homogeneity is $|\mathcal{I}(\Xi)|_{\mathfrak{s}} = \frac{1}{2} - \delta$.

$$\Pi_x^\varepsilon \mathcal{I}(\Xi\mathcal{I}(\Xi))(y) = G * (\xi_\varepsilon(G * \xi_\varepsilon))(y) - G * (\xi_\varepsilon(G * \xi_\varepsilon))(x)$$

and the homogeneity is $|\mathcal{I}(\Xi\mathcal{I}(\Xi))|_{\mathfrak{s}} = 1 - \delta$.

Examples

Setting $h := G * (\xi_\varepsilon(G * (\xi_\varepsilon(G * \xi_\varepsilon))))$,

$$\Pi_x^\varepsilon \mathcal{I}(\Xi\mathcal{I}(\Xi\mathcal{I}(\Xi)))(y) = h(y) - h(x) - \partial_{x_1} h(x)\,(y_1 - x_1)$$

and the homogeneity is $\frac{3}{2} - \delta$.

$$\begin{aligned}\Pi_x \mathcal{I}(X_1\Xi)(y) &= \\ &= \int (G(y-z) - G(x-z) - \partial_{x_1} G(x-z)\,(y_1 - x_1))\cdot \\ &\quad \cdot (z_1 - x_1)\,\xi_\varepsilon(z)\,\mathrm{d}z\end{aligned}$$

and the homogeneity is $|\mathcal{I}(X\Xi)|_{\mathfrak{s}} = \frac{3}{2} - \delta$.

Examples

For $\tau = \mathcal{I}(\Xi\mathcal{I}(\Xi\mathcal{I}(\Xi)))$ the homogeneity is $\frac{3}{2} - \delta$ and (dropping ε)

$$
\begin{aligned}
&\Pi_x^\varepsilon \mathcal{I}(\Xi\mathcal{I}(\Xi\mathcal{I}(\Xi)))(y) \\
&= G * (\xi G * (\xi G * \xi))(y) - G * (\xi G * (\xi G * \xi))(x) \\
&+ (y_1 - x_1)(\partial_{x_1} G * \xi(x)(G * (\xi G * \xi)(x) - G * \xi(x)^2)) \\
&+ (y_1 - x_1)\partial_{x_1} G * (\xi G * \xi)(x) G * \xi(x) \\
&- (y_1 - x_1)\partial_{x_1} G * (\xi G * (\xi G * \xi))(x) \\
&- G * \xi(x) G * (\xi G * \xi)(y) + G * \xi(x)^2 G * \xi(y) \\
&- G * (\xi G * \xi)(x) G * \xi(y) - G * \xi(x)^3 \\
&+ 2G * \xi(x) G * (\xi G * \xi)(x)
\end{aligned}
$$

Examples

For instance

$$\Gamma^{\varepsilon}_{xz}\mathcal{I}(\Xi) = \mathcal{I}(\Xi) + (G * \xi_\varepsilon(x) - G * \xi_\varepsilon(z))\mathbf{1}$$

and indeed setting $h := (G * \xi_\varepsilon)$

$$\begin{aligned}\Pi^{\varepsilon}_{x}\Gamma^{\varepsilon}_{xz}\mathcal{I}(\Xi)(y) =& h(y) - h(x) + h(x) - h(z) = h(y) - h(z) \\ =& \Pi^{\varepsilon}_{z}\mathcal{I}(\Xi)(y).\end{aligned}$$

Another example:

$$\begin{aligned}\Gamma^{\varepsilon}_{xz}\mathcal{I}(\Xi\mathcal{I}(\Xi)) =& (G * (\xi_\varepsilon(G * \xi_\varepsilon))(x) - G * (\xi_\varepsilon(G * \xi_\varepsilon))(z) \\ &+ (G * \xi_\varepsilon(z) - G * \xi_\varepsilon(x))G * \xi_\varepsilon(z))\mathbf{1} \\ &+ (G * \xi_\varepsilon(x) - G * \xi_\varepsilon(z))\,\mathcal{I}(\Xi) \\ &+ \mathcal{I}(\Xi\mathcal{I}(\Xi))\end{aligned}$$

Symbols with negative homogeneity

At the beginning we said that we would find, under a *subcriticality* assumption on the SPDE, a representation

$$u_\varepsilon = \mathscr{F}(g_1(\xi_\varepsilon), \ldots, g_k(\xi_\varepsilon))$$

where $\mathscr{F}$ is a continuous functional and $\{g_1, \ldots, g_k\}$ are explicit polynomial functionals of the smooth random function ξ_ε.

Well, the functionals $\{g_i(\xi_\varepsilon)\}$ are given by

$$\{\Pi^\varepsilon \tau, \ \tau \in \mathcal{F}_F, \ |\tau|_\mathfrak{s} < 0\}.$$

Each of these (finitely many) terms must be proved to converge separately. If this is done, convergence for all other symbols (trees) follows automatically by the theory.

However, before converging these terms must be renormalised.

Classical multiplication

Consider $f_i \in C^{\gamma_i}$, $i = 1, 2$, with $\gamma > 0$ and

$$f_i(y) = \sum_{j \le \gamma_i} a_i^j(x)\,(y-x)^j + r_i(x,y), \quad |r_i(x,y)| \le C(x)\|x-y\|^{\gamma_i}.$$

Pointwise multiplication gives, setting $\gamma = \gamma_1 \wedge \gamma_2$,

$$\begin{aligned} f_1 f_2(y) = & \sum_{j_1 \le \gamma_1} \sum_{j_2 \le \gamma_2} a_1^{j_1}(x)\, a_2^{j_2}(x)\,(y-x)^{j_1+j_2} \\ & + r_1(x,y)(f_2(y) - r_2(x,y)) + r_2(x,y)(f_1(y) - r_1(x,y)) + r_1 r_2(x,y) \\ & = \sum_{j \le \gamma} b^j(x)\,(y-x)^j + r(x,y), \end{aligned}$$

with $|r(x,y)| \le C(x)\|x-y\|^{\gamma}$.

After all, if f_1 and f_2 are in C^1 then $f_1 f_2$ is not necessarily in C^2 !

Multiplication of modelled distributions

We say that $F \in \mathcal{D}^\gamma_\eta$ if $F \in \mathcal{D}^\gamma$ and

$$F(x) = \sum_{j=1}^{N} a^j(x)\, \tau^j, \qquad \tau^j \in \mathcal{F}_F, \quad |\tau^j|_{\mathfrak{s}} \geq \eta.$$

Consider $F_i \in \mathcal{D}^{\gamma_i}_{\eta_i}$, $\gamma_i > 0$, $i = 1, 2$.

By the reconstruction theorem, $f_i := \mathcal{R}F_i \in \mathcal{S}'(\mathbb{R}^d)$ satisfies

$$f_i(y) = \sum_{j=1}^{N_i} a^j_i(x)\, \Pi_x \tau^j_i(y) + r_i(x, y)$$

where $|r_i(x, y)| \leq C(x)\|x - y\|^{\gamma_i}_{\mathfrak{s}}$ is a remainder.

Question: Can we define the product of f_1 and f_2?

Multiplication of Taylor sums

For $F_i \in \mathcal{D}^{\gamma_i}_{\eta_i}$ and $f_i := \mathcal{R}F_i \in \mathcal{S}'(\mathbb{R}^d)$, $j = 1, 2$,

$$F_i(x) = \sum_{j=1}^{N_i} a_i^j(x)\, \tau_i^j, \qquad f_i(y) = \sum_{j=1}^{N_i} a_i^j(x)\, \Pi_x \tau_i^j(y) + r_i(x,y),$$

$|r_i(x,y)| \leq C(x)\|x-y\|_{\mathfrak{s}}^{\gamma_i}$. Formal pointwise multiplication yields

$$(f_1 f_2)(y) = \sum_{j_1=1}^{N_1} \sum_{j_2=1}^{N_2} a_1^{j_1}(x) a_2^{j_2}(x)\, \Pi_x \tau_1^{j_1}(y) \Pi_x \tau_2^{j_2}(y)$$
$$+ r_1(x,y)\Pi_x F_2(x)(y) + r_2(x,y)\Pi_x F_1(x)(y) + r_1 r_2(x,y).$$

Multiplication of Taylor sums

$$(f_1 f_2)(y) = \sum_{j_1=1}^{N_1} \sum_{j_2=1}^{N_2} a_1^{j_1}(x) a_2^{j_2}(x)\, \Pi_x \tau_1^{j_1}(y) \Pi_x \tau_2^{j_2}(y)$$
$$+ r_1(x,y)\Pi_x F_2(x)(y) + r_2(x,y)\Pi_x F_1(x)(y) + r_1 r_2(x,y).$$

Now $|r_1(x,y)\Pi_x F_2(x)(y)| \le \|x-y\|_{\mathfrak{s}}^{\gamma_1+\eta_2}$. Therefore, setting $\gamma = (\gamma_1+\eta_2)\wedge(\gamma_2+\eta_1)$, we have

$$(f_1 f_2)(y) = \sum_{|\tau_i^1|+|\tau_i^2|<\gamma} a_i^1(x) a_i^2(x)\, \Pi_x \tau_1^{j_1}(y) \Pi_x \tau_2^{j_2}(y) + r(x,y)$$

with $|r(x,y)| \le C(x)\|x-y\|_{\mathfrak{s}}^{\gamma}$.

However $\Pi_x \tau_1^{j_1}(y)$ and $\Pi_x \tau_2^{j_2}(y)$ are distributions, so in general $\Pi_x \tau_1^{j_1}(y) \Pi_x \tau_2^{j_2}(y)$ is ill-defined.

Multiplication of Taylor sums

This suggests the following definition:

$$(F_1 F_2)(x) = \sum_{|\tau_i^1|_\mathfrak{s} + |\tau_i^2|_\mathfrak{s} < \gamma} a_i^1(x) a_i^2(x)\, \tau_i^1 \tau_i^2$$

and, by the reconstruction theorem, if $F_1 F_2 \in \mathcal{D}^\gamma$ and $\gamma > 0$ then $f_1 * f_2 := \mathcal{R}(F_1 F_2)$ satisfies

$$(f_1 * f_2)(y) = \sum_{|\tau_i^1|_\mathfrak{s} + |\tau_i^2|_\mathfrak{s} < \gamma} a_i^1(x) a_i^2(x)\, \Pi_x(\tau_1^{j_1} \tau_2^{j_2})(y) + r(x,y)$$

and can be defined as the product of f_1 and f_2.

If $\Pi_x \tau_1$ and $\Pi_x \tau_2$ are genuine distributions, their pointwise product is ill-defined in general and therefore Π_x can fail to be multiplicative. In the applications to SPDEs, it will be necessary to *renormalise* some of these products.

An abstract integration operator

We want to construct an operator $\mathcal{K} : \mathcal{D}^\gamma \to \mathcal{D}^{\gamma+2}$, $\gamma > 0$, $\gamma \notin \mathbb{N}$, s.t.

$$\mathcal{R}(\mathcal{K}V) = G * (\mathcal{R}V).$$

This is given by the following formula: if $V = \sum_\tau V_\tau(x)\,\tau$ then

$$\mathcal{K}V(x) = \sum_\tau V_\tau(x)\,\mathcal{I}(\tau)$$
$$+ \sum_{|k|_\mathfrak{s}<\gamma+2} \frac{X^k}{k!} \int_{\mathbb{R}^d} G^{(k)}(x-y)\left(\mathcal{R}V(\mathrm{d}y) - \Pi_x^\varepsilon Q^-_{|k|_\mathfrak{s}-2} V(\mathrm{d}y)\right)$$

where Q^-_η is the projection onto $\bigoplus_{\beta<\eta} T_\beta$.

An abstract differentiation operator

We want to construct an operator $\mathscr{D}_i : \mathcal{D}^\gamma \to \mathcal{D}^{\gamma-1}$, $\gamma > 0$, s.t.

$$\mathcal{R}(\mathscr{D}_i V) = \partial_i(\mathcal{R}V).$$

Let us set first $\mathscr{D}_i\tau$:

$$\mathscr{D}_i\mathbf{1} = 0, \qquad \mathscr{D}_i X_j = \delta_{ij}, \qquad \mathscr{D}_i\mathcal{I}_k(\tau) = \mathcal{I}_{k+\delta_i}(\tau),$$

$$\mathscr{D}_i(\tau\overline{\tau}) = \tau\mathscr{D}_i\overline{\tau} + \overline{\tau}\mathscr{D}_i\tau.$$

Then we see that

$$\Pi_x^\varepsilon \mathscr{D}_i\tau = \partial_i \Pi_x^\varepsilon \tau$$

and $\mathscr{D}_i : \mathcal{D}^\gamma \to \mathcal{D}^{\gamma-1}$ is constructed by linearity.

The lift of KPZ

We want to formulate the PDE in 1 space-dim.

$$\partial_t h = \Delta h + (\partial_x h)^2 + \xi_\varepsilon$$

as a fixed point in $\mathcal{D}^\gamma$

$$H = \mathcal{K}((\mathscr{D}H)^2 + \Xi).$$

What is the right γ? Note that H is a linear combination of powers of X and of $\mathcal{I}(\tau)$ with $|\tau|_\mathfrak{s} \geq -\frac{3}{2} - \delta$. Then $\mathscr{D}H \in \mathcal{D}^{\gamma-1}_{-\frac{1}{2}-\delta}$. Now $(\mathscr{D}H)^2 \in \mathcal{D}^{\gamma-1-\frac{1}{2}-\delta}$ so that we need $\gamma > \frac{3}{2} + \delta$ in order for the r.h.s. of the equation to be well defined. We choose $\gamma = \frac{3}{2} + 2\delta$.

The lift of KPZ

We expect

$$H = h\,\mathbf{1} + a\,\text{[tree]} + b\,\text{[tree]} + c\,X_1 + d\,\text{[tree]} + e\,\text{[tree]} \;\in \mathcal{D}^{\frac{3}{2}+2\delta}.$$

Then

$$\mathscr{D}H = a\,\text{[tree]} + b\,\text{[tree]} + c\,\mathbf{1} + d\,\text{[tree]} + e\,\text{[tree]} \;\in \mathcal{D}^{\frac{1}{2}+2\delta}$$

and

$$(\mathscr{D}H)^2 = a^2\,\text{[tree]} + 2ab\,\text{[tree]} + 2ac\,\text{[tree]} + b^2\,\text{[tree]} + 2bc\,\text{[tree]} + c^2\,\mathbf{1} \;\in \mathcal{D}^{\delta}$$

so that

$$\mathcal{K}((\mathscr{D}H)^2 + \Xi) = G * ((\partial h)^2 + \xi_\varepsilon)\,\mathbf{1} + \text{[tree]} + a^2\,\text{[tree]}$$
$$+\, G' * ((\partial h)^2 + \xi_\varepsilon - a^2\Pi^\varepsilon\text{[tree]})\,X_1 + 2ab\,\text{[tree]} + 2ac\,\text{[tree]} \quad \in \mathcal{D}^{\frac{3}{2}+2\delta}$$

We obtain that $a = b = 1$, $c = h'$, $d = 2$, $e = 2h'$, i.e.

$$
\begin{aligned}
H &= h\mathbf{1} + \<IXi> + \<I(I_1Xi)^2> + h' X_1 + 2\<I(I_1Xi(I_1Xi)^2)> + 2h' \<I(I_1Xi)> \\
&= h\mathbf{1} + \mathcal{I}(\Xi) + \mathcal{I}((\mathcal{I}_1(\Xi))^2) \\
&\qquad + h'X_1 + 2\mathcal{I}(\mathcal{I}_1(\Xi)(\mathcal{I}_1(\Xi))^2) + 2h'\mathcal{I}(\mathcal{I}_1(\Xi)).
\end{aligned}
$$

The theory shows that this fixed point in $\mathcal{D}^{\frac{3}{2}+2\delta}$ has (locally in time) a unique solution, which converges if $\Pi^\varepsilon\tau$ converges for all $|\tau|_\mathfrak{s} < \frac{3}{2} + \delta$.

However, as we have already announced several times, the unrenormalised family $\Pi^\varepsilon\tau$ fails to converge.

White noise

Let $(X, \mathscr{A}, m)$ be a σ-finite measure space with atomless measure m.

A *white noise* on $(X, \mathscr{A}, m)$ is a map $W : H := L^2(X, m) \mapsto L^2(\Omega, \mathbb{P})$, where $(\Omega, \mathscr{F}, \mathbb{P})$ is a probability space, such that

- $W(h) \in \mathscr{N}(0, \|h\|^2)$
- $\mathbb{E}(W(h_1)W(h_2)) = \langle h_1, h_2 \rangle$.

We can assume w.l.o.g. that $\mathscr{F} = \sigma(W(h), h \in H)$. Then a classical result says that there exists a natural isometry between $L^2(\Omega, \mathbb{P})$ and the symmetric Fock space

$$\mathcal{H} = \bigoplus_{k \geq 0} \mathcal{H}_k, \qquad \mathcal{H}_k \cong L^2_{\mathrm{sym}}(X^k, m^{\otimes k}) \cong H^{\otimes_s k}$$

$\mathcal{H}_k$ is the k-th homogeneous Wiener chaos, and $\bigoplus_{j \leq k} \mathcal{H}_j$ the k-th inhomogeneous Wiener chaos.

Multiplication of Wiener chaoses

We denote by I_k the natural isometry between $\mathcal{H}_k$ and $L^2_{\text{sym}}(X^k, m^{\otimes k})$. If $f \in L^2(X^k)$ then we denote by f_s its projection onto the symmetric subspace $L^2_{\text{sym}}(X^k)$ and $I_k(f) := I_k(f_s)$.

Then a formula (which we do not make more precise here) states that

$$I_\ell(f) I_m(g) \in \bigoplus_{j \le \ell + m} \mathcal{H}_j$$

with explicit projections on each homogeneous chaos.

An example: white noise on a point

If (X, m) is a point (0) with $m = \delta_0$, then $L^2(X, m) \cong \mathbb{R}$ and $L^2(\Omega, \mathbb{P}) \cong L^2(\mathbb{R}, \mathcal{N}(0, 1))$, $\mathcal{H}_k \cong \mathbb{R}H_k$, where $(H_k)_k$ are the Hermite polynomials:

$$H_0 = 1, \qquad H_{n+1}(x) = 2xH_n(x) - H_n'(x).$$

For instance $H_1(x) = x$,

$$H_2(x) = x^2 - 1, \quad H_3(x) = x^3 - 3x, \quad H_4(x) = x^4 - 6x^2 + 3$$

and the formula for the product is for $m \geq n$

$$H_m(x) * H_n(x) = 2^n n! \sum_{i=0}^{n} \binom{m}{n-i} \frac{H_{m-n+2i}(x)}{2^i i!}.$$

The *Wick product* is defined by $H_n \diamond H_m = H_{n+m}$.

White noise in $\mathbb{R}^d$

Let now (X, m) be $\mathbb{R}^d$ with the Lebesgue measure. We denote

$$W(h) =: \int_{\mathbb{R}^d} h(y)\,\xi(\,\mathrm{d}y).$$

Then ξ is a Gaussian field with the covariance function $\mathbb{E}(\xi(y)\xi(x)) = \delta(x-y)$.

It can be seen with the help of the Kolmogorov criterion that $x \mapsto W(\mathbb{1}_{[0,x]})$ is a.s. continuous, where $[0,x] = [0,x_1] \times \cdots \times [0,x_d]$. Then it can be seen that for $h \in \mathcal{S}(\mathbb{R}^d)$

$$W(h) = (-1)^d \int_{\mathbb{R}^d} \frac{\partial^d h}{\partial x_1 \cdots \partial x_d}\, W(\mathbb{1}_{[0,x]})\,\mathrm{d}x$$

and therefore ξ is a well-defined random Schwartz distribution.

Regularised white noise in $\mathbb{R}^d$

We set now

$$\xi_\varepsilon(x) := W(\rho_\varepsilon(x-\cdot)), \qquad \rho_\varepsilon(y) := \varepsilon^{-d-1}\rho\left(\frac{y_1}{\varepsilon}, \ldots, \frac{y_{d-1}}{\varepsilon}, \frac{y_d}{\varepsilon^2}\right)$$

and ρ is a smooth mollifier. Recall:

$$\Pi_x^\varepsilon X_i(y) = (y_i - x_i), \qquad \Pi_x^\varepsilon \Xi(y) = \xi_\varepsilon(y),$$

$$\Pi_x^\varepsilon(\tau_1 \cdots \tau_n)(y) = \prod_{i=1}^n \Pi_x^\varepsilon \tau_i(y),$$

$$\Pi_x^\varepsilon \mathcal{I}_k(\tau)(y) = (G^{(k)} * \Pi_x^\varepsilon \tau)(y) - \sum_{|i|_\mathfrak{s} < |\mathcal{I}_k(\tau)|_\mathfrak{s}} \frac{(y-x)^i}{i!} (G^{(i+k)} * \Pi_x^\varepsilon \tau)(x)$$

By the previous discussion, $\Pi^\varepsilon \tau$ is a r.v. in the k-th inhomogeneous Wiener chaos, with k the number of occurrences of the symbol Ξ in τ.

The Π^ε operator

Let us set

$$\Pi^\varepsilon X_i(y) = y_i, \qquad \Pi^\varepsilon \Xi(y) = \xi_\varepsilon(y),$$

$$\Pi^\varepsilon(\tau_1 \cdots \tau_n)(y) = \prod_{i=1}^{n} \Pi^\varepsilon \tau_i(y),$$

$$\Pi^\varepsilon \mathcal{I}_k(\tau)(y) = (G^{(k)} * \Pi^\varepsilon \tau)(y).$$

This operator is the "stationary" version of Π_x^ε.

Again, $\Pi^\varepsilon \tau$ is a r.v. in the k-th inhomogeneous Wiener chaos, with k the number of occurrences of the symbol Ξ in τ.

Divergence of the Π^ε operator

Let us consider for instance $\tau = \mathbb{V}$. Then

$$\Pi^\varepsilon \mathbb{V}(\varphi) = \int_{\mathbb{R}^2} \varphi(y)(\Pi^\varepsilon \mathbb{I})^2(y)\,\mathrm{d}y = \int_{\mathbb{R}^2} \varphi(y)(G' * \xi_\varepsilon)^2(y)\,\mathrm{d}y$$

which belongs to $\mathcal{H}_0 \oplus \mathcal{H}_2$. It is easy to see that

$$\mathbb{E}(\Pi^\varepsilon \mathbb{V}(\varphi)) = \int_{\mathbb{R}^2} \varphi(y)(G' * \rho_\varepsilon * \rho_\varepsilon * G')(0)\,\mathrm{d}y \approx \frac{1}{\varepsilon}.$$

Instead we have that $\mathrm{Var}(\Pi^\varepsilon \mathbb{V}(\varphi))$ remains bounded.

Renormalisation of Π^ε

We now introduce a linear $M_\varepsilon : T \mapsto T$ and $\hat{\Pi}^\varepsilon := \Pi^\varepsilon \circ M_\varepsilon$. However M_ε can not be arbitrary and must respect the structure we have built. In particular, we expect $\hat{\Pi}^\varepsilon$ to have the following recursive construction:

$$\hat{\Pi}^\varepsilon X_i(y) = y_i, \qquad \hat{\Pi}^\varepsilon \Xi(y) = \xi_\varepsilon(y),$$

$$\hat{\Pi}^\varepsilon(\tau_1 \cdots \tau_n)(y) = \prod_{i=1}^n \hat{\Pi}^\varepsilon \tau_i(y) - \hat{\Pi}^\varepsilon \left[L_\varepsilon(\tau_1 \cdots \tau_n)\right](y),$$

$$\hat{\Pi}^\varepsilon \mathcal{I}_k(\tau)(y) = (G^{(k)} * \hat{\Pi}^\varepsilon \tau)(y).$$

We are modifying the products.

This imposes several restrictions on L_ε and M_ε.

The $\hat{\Pi}_x$ operators

We then define

$$\hat{\Pi}^{\varepsilon}_x X_i(y) = (y_i - x_i), \qquad \hat{\Pi}^{\varepsilon}_x \Xi(y) = \xi_{\varepsilon}(y),$$

$$\hat{\Pi}^{\varepsilon}_x(\tau_1 \cdots \tau_n)(y) = \prod_{i=1}^{n} \hat{\Pi}^{\varepsilon}_x \tau_i(y) - \hat{\Pi}^{\varepsilon}_x \left[L_{\varepsilon}(\tau_1 \cdots \tau_n)\right](y),$$

$$\hat{\Pi}^{\varepsilon}_x \mathcal{I}_k(\tau)(y) = (G^{(k)} * \hat{\Pi}^{\varepsilon}_x \tau)(y) - \sum_{|i|_{\mathfrak{s}} < |\mathcal{I}_k(\tau)|_{\mathfrak{s}}} \frac{(y-x)^i}{i!} (G^{(i+k)} * \hat{\Pi}^{\varepsilon}_x \tau)(x)$$

We have in general

$$\hat{\Pi}^{\varepsilon}_x \tau(y) \neq \Pi^{\varepsilon}_x M_{\varepsilon} \tau(y), \qquad \hat{\Pi}^{\varepsilon}_x \tau(x) = \Pi^{\varepsilon}_x M_{\varepsilon} \tau(x).$$

These matrices are of the form $M = \exp(-\sum_{i=0}^{3} C_i L_i)$, where the generators L_i are determined by the following contraction rules:

L_0 :  $\mapsto \mathbf{1}$, L_1 :  $\mapsto \mathbf{1}$, L_2 :  $\mapsto \mathbf{1}$ L_3 :  $\mapsto \mathbf{1}$.

This should be understood in the sense that if τ is an arbitrary formal expression, then $L_0\tau$ is the sum of all formal expressions obtained from τ by performing a substitution of the type  $\mapsto \mathbf{1}$. For example, one has

L_0  $= 2$, L_0  $= 2$  $+$  ,

etc.

Renormalisation of KPZ

Theorem
Let $M_\varepsilon = \exp(-\sum_{i=0}^3 C_i^\varepsilon L_i)$ be as above and let $(\hat{\Pi}^\varepsilon, \hat{\Gamma}^\varepsilon)$ be the corresponding renormalised model. Let furthermore H be the solution to the lifted KPZ

$$H = \mathcal{K}((\mathscr{D}H)^2 + \Xi)$$

with respect to this model. Then, the function $\hat{h}_\varepsilon = \mathcal{R}H$ solves the equation

$$\partial_t \hat{h}_\varepsilon = \partial_x^2 \hat{h}_\varepsilon + (\partial_x \hat{h}_\varepsilon)^2 - 4C_0^\varepsilon \, \partial_x \hat{h}_\varepsilon + \xi_\varepsilon - (C_1^\varepsilon + C_2^\varepsilon + 4C_3^\varepsilon) \ .$$

Proof

$$\mathscr{D}H = \<1> + \<2> + h'\,\mathbf{1} + 2\<3> + 2h'\<1a> \quad \in \mathcal{D}^{\frac{1}{2}+2\delta}$$

$$(\mathscr{D}H)^2 + \Xi = \Xi + \<11> + 2\<12> + 2h'\,\<1> + \<22> + 4\<23> + 2h'\,\<2> + 4h'\,\<1b> + (h')^2\,\mathbf{1}\ .$$

We recall the rules $L_0\<1b> = L_1\<11> = L_2\<22> = L_3\<23> = \mathbf{1}$,

$$L_0\<3> = 2\<1a}\ , \quad L_0\<12> = 2\<1}\ , \quad L_0\<23> = 2\<1b> + \<2>\ .$$

Then $M_\varepsilon\mathscr{D}H = \mathscr{D}H - 4C_0^\varepsilon\<1a>$, $(M_\varepsilon\mathscr{D}H)^2 = (\mathscr{D}H)^2 - 8C_0^\varepsilon\<1b>$

$$\begin{aligned} &M_\varepsilon\big((\mathscr{D}H)^2 + \Xi\big) \\ &= (\mathscr{D}H)^2 + \Xi - C_0^\varepsilon\big(4\<1> + 4\<2> + 8\<1b> + 4h'\,\mathbf{1}\big) - C_1^\varepsilon - C_2^\varepsilon - 4C_3^\varepsilon \\ &= (M_\varepsilon\mathscr{D}H)^2 + \Xi - 4C_0^\varepsilon\, M_\varepsilon\mathscr{D}H - (C_1^\varepsilon + C_2^\varepsilon + 4C_3^\varepsilon)\ . \end{aligned}$$

Then $\hat{h}_\varepsilon = G * \big((\partial_x\hat{h}_\varepsilon)^2 + \xi_\varepsilon - 4C_0^\varepsilon\,\partial_x\hat{h}_\varepsilon - (C_1^\varepsilon + C_2^\varepsilon + 4C_3^\varepsilon)\big)$.

Convergence of the model

Theorem. If we choose

$$C_0^\varepsilon = 0\ , \quad C_1^\varepsilon = \frac{c_1}{\varepsilon}\ , \quad C_2^\varepsilon = 4c_2 \log \varepsilon + c_3\ , \quad C_3^\varepsilon = -c_2 \log \varepsilon + c_4\ .$$

then the renormalised model $(\hat{\Pi}^\varepsilon, \hat{\Gamma}^\varepsilon)$ converges.

In the limit, the solution is parametrised by $c_3 + 4c_4$.

This is a general fact: there is a *renormalisation group* $\mathfrak{R}$ such that $M_\varepsilon \in \mathfrak{R}$. If M_ε and N_ε are different renormalisation maps, then $M_\varepsilon N_\varepsilon^{-1}$ converges to $R \in \mathfrak{R}$ and the two limit solution differ by the action of R.

The group $\mathfrak{R}$ acts on the set of renormalised solutions.